THE MATHEMATICAL GAZETTE

EDITED FOR THE MATHEMATICAL ASSOCIATION BY

R. L. GOODSTEIN

WITH THE ASSISTANCE OF

H. M. CUNDY K. M. SOWDEN

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- De Sectionibus Conicis Tractatus.
- Arithmetica Infinitorum: sive de Curvilineorum quadraturâ, &c.
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THE

MATHEMATICAL GAZETTE

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FIFTY YEARS OF CHANGE

A. W. SIDDONS

There must be many teachers of mathematics today who do not realise the great changes, both in what was taught and in methods of teaching, that were made in the early days of this century and the part that the Mathematical Association took in making them.

To understand what those changes are and to appreciate the benefits that they have conferred on teachers and pupils, it is desirable to know something of the history of the Mathematical Association.

In 1871 the Association for the Improvement of Geometrical Teaching was founded at the suggestion of my old mathematical master, Rawdon Levett. In 1897 its name was changed to the Mathematical Association.

The A.I.G.T. and the M.A. up to about 1912 may be said to have consisted of Public and Grammar School masters with a fair sprinkling of more advanced teachers who were interested in what was taught in the schools concerned. In spite of the resultant narrowness it had considerable influence on the work of other types of schools. In 1902 the first Teaching Committee was formed and reports were published on Geometry, Arithmetic and Algebra. In 1907 a report was published on mathematical teaching in Preparatory Schools. In 1912 committees were formed for dealing with mathematics in (i) Public Schools, (ii) Other Secondary Schools for boys, (iii) Girls' Schools. In recent years the scope of the work has been increased to include mathematics in all types of schools.

The first number of the *Mathematical Gazette* appeared in 1894. On the cover of that number it said, " We intend to keep strictly to Elementary Mathematics : while not excluding Differential and Integral Calculus, our columns will, as a rule, be devoted to such school subjects as Arithmetic, Algebra, Geometry, Trigonometry and Mechanics." This limitation gradually disappeared and the Gazette developed a great reputation for its reviews of books, both elementary and advanced, and articles on advanced subjects have figured largely in its pages. In recent years the number of articles dealing with elementary topics has been comparatively small, in spite of frequent appeals by the Editor ; but some items of interest to the main body of teachers have appeared in the form of Notes. At the moment steps are being taken in hopes of providing many more articles that deal with elementary topics and that will be of interest to the main body of teachers in all types of schools.

The main object of the A.I.G.T. was to free teachers and pupils from the

bondage of Euclid, whose treatise was intended for the use of university students and is quite unsuitable for introducing school children to geometry. In the early days of the A.I.G.T., Euclid's proofs were required in examinations and no other proofs were allowed. First of all the A.I.G.T., after very careful work, produced a syllabus of geometry and in 1886 a textbook of geometry was published. Even these would hardly be considered suitable for beginners today. Both the Syllabus and the Textbook were very favourably received, but Cambridge would not move from its position until 1888 when it announced that proofs other than Euclid's would be accepted *so long as they did not violate Euclid's order*.

At that time and even into the early days of the present century, in many schools, all the geometry that was done was the learning of Euclid's proofs. In many cases it was done unintelligently. To take an extreme example here is a story told me years ago by the then Master of Jesus College, Cambridge. Up till 1850 no man was allowed to take the classical tripos at Cambridge until he had passed the mathematical tripos. A brilliant classic who was weak at mathematics went to a coach and asked him to choose 20 propositions that were likely to be set in the mathematical tripos and he said " I will learn them by heart ". Later the number was reduced to 10. After the examination the delighted undergraduate rushed up to his coach's room and said " I am through ; I got 8 of the propositions you chose and I got them all right to a comma ". As an afterthought he added " I am not sure that I put the right letters at the right corners, but I suppose that does not matter ".

I was a boy at King Edward's School, Birmingham, at which the mathematical teaching was probably as good as at any school in the country. We did many riders from the start, but there were very few riders in our textbook and we generally had to take them down when they were set.

When I started teaching at Harrow in 1899, in the middle divisions of the upper school I found many boys who professed never to have done a rider—that does not mean that there were boys in higher divisions who had not done many riders, but it shows how low was the standard of the majority.

At the beginning of the present century Professor John Perry started an agitation against existing methods of teaching mathematics and the matter that was being taught. At the Glasgow meeting of the British Association in 1901 he opened a discussion on this subject and put forward a schedule of work. 15 or 16 people spoke after Professor Perry—only one of them a working schoolmaster. After the meeting, the British Association appointed a committee to go into the matter ; on that committee, if I remember rightly, there was only one working schoolmaster and he taught science, not mathematics.

However, Professor A. R. Forsyth, who was chairman, invited Charles Godfrey, the senior mathematical master at Winchester, to write a letter to the committee suggesting reforms that could be made. This letter was signed by 23 schoolmasters and was reprinted in *Nature* and in the *Mathematical Gazette* of January 1902.

In January 1902 the M.A. appointed its first Teaching Committee. The committee included several men who had been active members of the A.I.G.T. in its later days ; it also included 4 or 5 schoolmasters who were under 30 years of age, and later other youngish schoolmasters were coopted. In less than a year reports were published on Geometry, Arithmetic and Algebra. These reports recommended moderate changes with which it was hoped most teachers would agree, so that examining bodies might be almost forced to make the changes desired.

In December 1902 Cambridge appointed a Syndicate to report on University pass examinations and, in particular, what concerned school teachers most, the " Little go ". This Syndicate recommended that Euclid's order should no longer be required, but that " Any proof of a proposition should be

accepted which appeared to form part of a systematic treatment of the subject ". The report was accepted by the Senate and Euclid's order no longer dominated the teaching of Geometry. The first examination under the new regulations was held in March 1904.

The importance of this change can hardly be exaggerated, because the requirements of the University entrance examinations dominated the teaching of Geometry in the schools, and, until the Universities moved, the schools could do little. It was probably the greatest reform ever made in the teaching of any subject; it was the forerunner of many reforms in mathematics and in other subjects.

After this historical sketch, I will now consider in detail some of the changes that have been made. As my main experience has been in Public Schools, I will deal with those, but most of what I shall say will apply equally well to other schools, both boys' and girls'. I shall quote freely from the letter of the 23 schoolmasters which shows fairly well the aims of the reformers.

That letter said:

" As regards Geometry, we are of opinion that the most practical direction for reform is towards a wide extension of accurate drawing and measuring in the Geometry lesson. This work is found to be easy and to interest boys; while many teachers believe that it leads to a logical habit of mind more gently and naturally than does a sudden introduction of a deductive system."

As to this work of drawing and measuring, very little was done until after the beginning of the present century. In his presidential address to the M.A. in 1947, Mr. W. F. Bushell said that in his last term at Charterhouse, in 1903, he was introduced to a protractor for the first time. As soon as Euclid was dethroned, this work was taken up in most schools; but the spirit of the reformers was often misunderstood, in many cases it was overdone and did not lead to anything useful. However, in 1909 the Board of Education issued the famous circular 711 (mainly the work of Mr. W. C. Fletcher) which was the first authoritative statement on the subject and put out the aims of the reformers in a very clear form.

Circular 711 divided the teaching of geometry into three stages. The following explains what those stages are, but not necessarily in the words of the circular.

First stage. To gain familiarity with and clearness of perception of fundamental geometrical concepts by means of observation of the common facts of life and practical work.

The concepts to be dealt with are solid, surface, line, point, volume, area, length, direction, angle, parallelism.

Definitions should be avoided, but emphasis should be laid on the right use of words.

Second stage. Discovery of the fundamental facts of geometry, by experiment and intuition; including the facts relating to angles at a point, parallels, angles of a triangle and polygon, congruent triangles.

When discovered each fact or group of facts should be followed by numerical examples and easy riders intended to illustrate and drive home the facts discovered. It is well worth while memorising in words the facts discovered.

In the course of this stage, the pupil, besides becoming familiar with the fundamental facts, should learn the accurate use of instruments and the elementary ideas of logical argument.

Though circular 711 does not mention it, there is much to be said for including in this stage a course on similar figures. Pupils, from scale drawing and maps, already have a good deal of subconscious knowledge about similar figures, and there are easy riders on the subject.

Third stage. A logical course of deductive geometry, based on the fundamental concepts and facts developed in the previous stages.

No attempt should be made at proving the fundamental facts about angles at a point, parallels and the cases of congruence: these should be regarded as axioms on which the deductive geometry is based. Today in most examinations proofs of these theorems are not required.

As has been said above, the old work in geometry consisted largely of learning the proofs of Euclid's propositions. Today it may be said that at first the teaching aims at bringing into the conscious plane many facts of which subconsciously the pupil is already aware; then developing these ideas and gradually giving geometrical power and some power of intuition. A knowledge of the theorems should be led up to by doing suitable riders, and by experiment and measurement whenever these help. Then, as a final stage, there is the development of a systematic course.

In the systematic course related facts should be grouped together. The one thing that the pupil has to remember is what is the fundamental fact in each group.

An able teacher will find today that geometry is not a difficult subject: in fact it can be attractive to most pupils.

It has been said that "Geometry is the most powerful weapon in Mathematics, but it requires the skill of a Newton to use it". No doubt the author of this epigram had in mind Newton's great book "the Principia" which was largely geometrical, but modern methods do not bear out this description of its difficulty today, at all events as far as the elementary stages are concerned.

Circular 711 also dealt with graphs, so I will consider them next. Before the beginning of this century mathematical teaching did not deal with graphs at all: paper ruled in inches and tenths was practically unknown in the mathematical class-room. Pupils might come across them in connexion with science, but otherwise not at all. Professor Perry in his British Association talk in 1901 spoke of the use of squared paper. Chapters on graphs were introduced into textbooks, but they were generally mere watered-down coordinate geometry: many graphs of straight lines were drawn and simultaneous linear equations solved by graphs—a process which must have seemed useless to any intelligent pupil. Contrast all that with the position today. Graphs are used everywhere: we see them in the newspapers and in books of all sorts: at least their appearance is familiar to our pupils.

The first step in the introduction of graphical work should be plotting statistics, using material with which the pupil is already familiar, for instance, a chart showing a child's weight at different ages, a temperature chart, or a barograph chart, possibly even a travel graph. The next thing is to interpret graphs: to see what can be learnt from a graph such as one showing the cost of a commodity at different dates. When was the cost highest, when lowest, when was it increasing, when was it increasing fastest?

The next step should be to consider graphs of functions; here it is much better to start with curved graphs instead of with straight lines. Circular 711 considered the following:

It is 120 miles from London to Bristol. What will be the average speed of trains covering the distance in 2, 3, 4, ... hours? Plot the results.

"The question may be carried further by supposing the time of transit further altered, and as we get away from practical speeds of trains we can use the speed of sound or light at one end, the speed of cyclist, coach, wagon, pedestrian at the other, to suggest the idea of the complete curve. We thus arrive at the utility of the algebraic expression $\dfrac{120}{x}$, together with the graphic exhibition of all its values, including incidentally in a simple and effective fashion the new concept of infinity and the new mathematical concept of zero."

Circular 711 then says that pupils may go on to plot any function and suggests $(x-2)(x-4)$ as a start. After plotting that, they may go on to solve

equations such as $(x-2)(x-4)=5$. Then some work can be done on straight line graphs and they can see that the solution of two simultaneous equations such as $y=(x-2)(x-4)$ and $y=2x-3$ is found by considering the point of intersection of the two graphs.

Perhaps it might be added that even today graphs are sometimes mis-used by less skilled teachers. A certain imagination is necessary to get the best out of them: they are more important as a preparation for the Calculus than as a beginning of Analytical Geometry.

Arithmetic, as I remember it as a boy and in my early teaching days, was a series of long sums and many rules. My old mathematical master, who was a rebel against all this, used to say that there are five rules in arithmetic: addition, subtraction, multiplication, division and common sense. Would that I had had more teaching from him than from some of the masters that taught me! But they were hardly to blame, for external examinations encouraged such treatment of the subject.

As to the beginning of the subject, getting the idea of number and learning tables and accurate computation, I will only say that I sometimes wonder whether pupils today are as accurate in plain straightforward computation as they were fifty years ago. After all, accuracy in computation consists mainly in being able (i) to add a number from 1 to 9 to another number between 1 and 99; (ii) to know the multiplication tables from 2 up to 12 times. This knowledge once acquired by means of viva voce practice and short sums, accuracy in longer sums is largely a matter of concentration—that should come gradually as a pupil grows older.

Rules were given for everything, generally with no explanation. Even for placing the decimal point in a multiplication sum (by adding the number of decimal places in multiplier and multiplicand) pupils were given the rule without explanation. [The Editor would like to ask modern teachers whether they are successful in explaining this rule—which is recommended in the M.A.'s *Report on the Teaching of Arithmetic*—to children of 11 and 12.]

Now for a few of the things that were done but are not done now.

Much time was spent on long money sums and on reduction; this reduction was often absurd as for instance reducing tons, cwts, . . . to ounces or miles to inches.

Many obsolete tables or tables concerned with special trades (e.g. Troy weight) were used.

In areas it was not often pointed out that 1 sq. ft. $=12^2$ sq. in., etc.

Rules for H.C.F. and L.C.M. were taught without any explanation, and little or no stress was laid on finding them by means of prime factors.

Complex fractions with several " stories " often took up much time.

Weeks were spent on recurring decimals and reducing them to vulgar fractions; by rule of course.

Alligation (the rule of mixtures), Present Worth, Banker's and True Discount, Square and Cube roots, etc., were all taught by rule. Some of these have disappeared from school textbooks; others, if done at all, are done by common sense.

There were chapters on Problems. Instead of being treated as problems, they were classified and rules were given for their solution.

The use of x and a little algebra, which would have made many problems easy by common sense were not allowed in Arithmetic. In my school days, if an x were used in the arithmetic paper of an outside examination, we might assume that no marks would be given for the answer. This difficulty could sometimes be dodged by writing " the unknown number " instead of x.

Logarithms were never used in Arithmetic.

Rough checks were seldom used, and too many figures were often expected in answers. I do not remember any reference to the number of significant

figures that could be trusted in an answer, even when π had been taken as $3\frac{1}{7}$.

There is one rule in Arithmetic that at one time I would have said was worth learning without explanation ; that is the rule for finding a square root. But Sir Percy Nunn suggested a geometrical approach to the subject which makes at least some part of the rules intelligible and reasonable to a child.

Today Arithmetic teaching is much more rational : instead of pupils being given rules (with little or no explanation) new processes are introduced by viva voce examples with numbers that involve little manipulation. It is much more important that a pupil should have an understanding of the idea that underlies a new process than that he should be able to apply a rule ; and the understanding can be acquired better from simple cases that involve little manipulation.

Today stress is laid on rough checks and rough estimates before a sum is worked ; these, besides being useful, often bring out the principles that are being used better than the actual sum. Stress is also laid on significant figures and the degree of accuracy of an answer. Altogether common sense plays a much greater part in the work.

Now let me turn to Algebra. There were a few pupils who quite quickly came to appreciate the beginnings of Euclid and to get some intellectual pleasure out of it—very few I grant ; but I cannot imagine anyone becoming interested in the beginnings of Algebra as taught 50 or 60 years ago.

I started Euclid and Algebra at the age of 13, though most boys started much earlier ; my recollection of my start in Algebra is that I spent a term finding the values of expressions such as ab^2c^3 for given values of a, b and c and then going on to the four rules, addition etc. The expressions we had to evaluate were not interesting formulas, and the expressions we met for the four rules, were equally meaningless ; and the rules seemed rather arbitrary. I suppose I was a good little boy and got a good many sums right, but it meant nothing to me and roused not the slightest interest. For the next two terms I was up to Mr. Levett, to whom I was afterwards up for my last two years at school. What a different thing Algebra became under him : we got on to equations and problems and then there was some interest ; we began to see some use in Algebra and we learnt to use our common sense Other masters just told boys to move a term from one side of an equation to the other side and change its sign, a rule which many teachers gave without explanation.

Later, under other masters, we went on to a lot of academic stuff which made little appeal : I remember much work on sums and products of roots of quadratic equations which seemed to have no use until terms later we began analytical geometry. Of course we spent much time on fractions and finding H.C.F.'s by a rule the essentials of which were never pointed out, or we did not grasp them. In dealing with surds we were hardly ever required to evaluate such expressions as $5 \pm \sqrt{3}$.

Some boys, after many terms of algebra, reached the study of logarithms, but the work was mainly academic. A friend of those days writes to me " No-one taught us the meaning of logarithms. We simply obeyed rules. Today by means of the index notation we try to explain the principles involved ". If we used logarithms at all, we only had Chamber's Seven-figure tables ; it is far easier to get practice in using logarithms from Four-figure tables, but I never saw such tables in my school days.

All the stress in our early work on Algebra seemed to be laid on accurate manipulation of expressions much more complicated than any we were likely to meet until the time we became mathematical specialists, which the majority of us never would be. It may well be asked whether the future mathematical specialist does not suffer by not acquiring this manipulative skill early. I do not think that he does : when he becomes a specialist he can acquire that skill much more quickly because he is not handicapped by doing that work in

a class the majority of whom are uninterested, and, further than that, he begins to see the value of it.

It is easy to find contemporary allusions to the teaching of Algebra in those days, and hence to realise how unsuccessful it was. For instance, the late Dr. Alington, formerly Headmaster of Eton, went as a boy to Marlborough in 1886. In "Things ancient and modern" he writes—"Mathematics were pursued to the level of the Higher Certificate Examination, and I look back with loathing on many a wasted hour . . . I will only say that the reason why I learnt Algebra is as obscure to me today as it was in the days when I suffered from it, and that those of my instructors whom I have had the chance of consulting are unable to throw any light on the question." Or I could quote Dr. A. N. Whitehead, who cannot be suspected of any prejudice against the subject; he speaks of "scraps of gibberish taught to boys in their schooldays in the name of Algebra".

Contrast what I have said of the old days with the beginnings of Algebra today: a pupil starts with a simple problem leading to a very simple equation or with formulae within his experience. My own preference is for the problem-equation start. In the course of a problem and its resulting equation a little simple manipulation arises; that type of manipulation is mastered and practised. Then another problem is tackled that leads to more manipulation and that too is mastered. In this way power of manipulation seems to be useful and is gradually acquired. Algebra soon begins to have some meaning and attraction for the pupil.

Of graphs I have already spoken. I never came across a graph in my schooldays, and none of my first pupils had. I will not say more of them here except that when graphing a function such as $(x-1)(x-3)$ many useful lessons about the multiplication of negative numbers can be driven home. One other point arises when plotting points on a graph: if one point is obviously not on a smooth curve through the other points, the pupil sees at once that he must have made a mistake and can correct it without being told of it.

Before I leave Algebra, I should like to point out how little Algebra is needed for a pupil to start Calculus; he can learn to differentiate, say x, x^2, x^3 and $\frac{1}{x}$, to integrate the corresponding expressions and to apply these to a great variety of practical problems. Thus he can open up a vista of what mathematics can do. But I am not advocating that he should go on to this work as soon as he has acquired the necessary algebra; there are other things he should do first.

I have talked about Geometry, Graphs, Arithmetic and Algebra; but what about other subjects? Fifty odd years ago in most schools some, but far from all, boys started Trigonometry, fewer still did some Mechanics and very few did any Calculus. And what was the work that was done in those subjects?

Few boys used Trigonometry for practical questions on heights, distances, etc. If they did, it was generally limited to angles of 30°, 45°, 60° or perhaps 15° and 75°; also the only tables used were Chambers Seven-figure tables. I never met any Four-figure tables till my fourth year at Cambridge when I was working for the Science Tripos. They seemed to be unknown to mathematical masters when I started teaching.

At the start of Trigonometry six definitions were usually fired at the pupil; he went on to compound angles and many trigonometrical identities. We spent hours on these. Little interest was aroused by this, except for the few who later specialised in mathematics.

Today a pupil starts with one or two definitions and with four-figure tables (or perhaps even three-figure) does simple examples on heights, distances, etc. The majority of pupils take to this easily by the time they are 14 or earlier. It shows some of the uses of Trigonometry and arouses interest at once. It

also drills him in the use of Four-figure Tables and leads him on to take interest in the more academic parts of the subject.

Mechanics, like Geometry, is a subject in which a child has a considerable subconscious knowledge before he takes it up in school. That subconscious knowledge needs to be developed and systematised ; I do not remember that this was done when I was at school. We were plunged into the parallelogram of forces and resolution without any appeal to our experience. We were told about the triangle of forces but did not use it to work out little problems by graphical methods. We might well have started with the law of moments ; appeal to our experience would have helped there. Historically the law of moments comes before the parallelogram or triangle of forces. In dynamics we learnt algebraic proofs of $s = ut + \frac{1}{2}at^2$, etc., proofs which were much more easily done by calculus. Then we even tackled projectiles, with angles of projection of course of 30°, 45° or 60°. Little stress was laid for a long time on momentum and energy.

In our school days we never did any practical experiment in mechanics and never saw any done by our masters. As we progressed further with mechanics we met many examples most of which consisted of a small percentage of mechanics and a large amount of algebraic manipulation.

I remember when I started teaching Calculus to some fairly bright boys of about 14 or 15, a classical colleague, who had been quite good at mathematics as a schoolboy said to me " Surely you are mad to start such boys on Calculus ; is not that the crown of school mathematics and only suitable for the very few? " I agree that Calculus as taught 50 or 60 years ago is not suitable for boys in the middle of the school. At the end of the last century Calculus questions were beginning to appear in Oxford and Cambridge papers for entrance scholarships, but what was the Calculus taught then? We worked for a few terms at differentiating all sorts of functions ; this was the main work and many pupils never got further. Later some of us got on to applications and so learnt some of the uses of the subject and began to appreciate its power. A few boys started Integral Calculus ; but here again there was a long grind at mere integration before any applications and so interest and appreciation of its power did not come for a long time.

Contrast this with conditions today : then only very few learnt Calculus ; today, or perhaps very soon, we may find half the candidates for the General Certificate of Education at ordinary level doing a little Calculus.

There are three fundamental ideas in elementary Calculus :

(i) The idea of the gradient function of a graph and so of differentiation.
(ii) The idea of finding an area by a process of anti-differentiation.
(iii) The idea of finding an area as the limit of a process of summation and so of integration.

After seeing (ii) and (iii) the pupil gets the idea of integration by means of anti-differentiation.

All these ideas can be developed by differentiating x, x^2, x^3, $\frac{1}{x}$ and integrating the corresponding functions. In the course of this work pupils apply the ideas to many interesting problems and so even those who go no further with mathematics learn some of the uses of Calculus ; they learn to appreciate its power and so a vista is opened up to them. As I pointed out before, the amount of Algebra needed to do this work is very little.

Those who are to go further will soon ask about differentiating and integrating other functions such as the trigonometrical functions and later e^x.

In most of the above I have considered work which is within the range of pupils of average ability who may never go further with mathematics. That work can be made attractive. The great thing is to maintain the interest

and that is best done by introducing new topics, so far as possible, with practical problems so that the use of the work is soon evident.

But what about the future specialist? He loses nothing by first going through a course such as I have in mind. The more technical knowledge and skill that he needs he can acquire very quickly and willingly. Even in his case there is gain in making the work interesting; the modern teacher tries to show him where his work is leading. In this way the pupil may acquire some skill in and see the applications of pieces of work before he investigates the foundations on which they rest. If he is anything of a mathematician, when he has some idea of the goal he is seeking, he will want to make the foundations solid. It should be remembered that this is the way Mathematics has often developed historically.

To sum up it may be said that 50 or 60 years ago mathematics was presented to pupils in the cold, systematic, logical form in which the academic mathematician might ultimately pigeon-hole it in his mind. This form was quite unsuitable for beginners and, though later it might appeal to the real mathematician, it would never appeal to the vast majority of pupils; certainly it is true to say that the education of 95 per cent. of pupils was being spoilt for the very doubtful benefit of the remaining 5 per cent.

In so short an essay it has been impossible to do more than indicate some of the immense changes, not only in what is taught in the Mathematical classroom, but also in the method of presentation. I do not delude myself into the belief that we have now reached finality. Probably we are still on the upward path and learning slowly to teach better. It is Izaak Walton, in the *Compleat Angler*, who says that " Angling may be said to be so like the mathematics that it can never be fully learnt ". Probably the best method of presentation can never be fully learnt, but I do claim that there has been an enormous advance during my lifetime. It may well be that a member of our Association, writing in the year 2000, will look back with pitying scorn on all he had to endure in his schooldays in the middle of this century. Indeed I hope this may be so, thus indicating yet another advance. But I do claim that my generation has done something for the improvement of the teaching of Mathematics.

I am much indebted to Mr. W. F. Bushell who has read through my manuscript and made many helpful suggestions which I have adopted.

A. W. S.

GLEANINGS FAR AND NEAR

1859. But all these and other devices may be thwarted by the curl of the opponent's stone, which, as it dies, may turn almost at right or left angles and defeat the best laid schemes.—*Chambers's Encyclopedia* (1950), II, *s.v.* Curling. [Per Mr. J. C. W. De la Bere.]

1860. The deal involves the bulk of the triangular area bounded by Temple Row, Bull Street, Cherry Street and Corporation Street.—*Birmingham Post*, September 16, 1955. [Per Mr. J. J. Parker.]

AN INTRODUCTION TO THE MATHEMATICAL THEORY OF INFORMATION

B. C. Brookes

Introduction

Information Theory was developed to provide electrical engineers with a calculus for comparing telecommunications systems as transmitters of "information"—an ambiguous word used here in a technical sense to be defined. Since 1948 the theory has been rapidly developed and widely applied, but the results of its application to telecommunication engineering have been disappointing in some respects. Although it has been applied indiscriminately to all kinds of communication systems including language, it has nevertheless stimulated new ways of thinking about the storage and transmission of every kind of "information" in the most general sense of that word. It has attracted the attention of workers in fields as diverse as neurology and librarianship, statistical mechanics and psychology, cryptography and the study of social insects. Its terminology and concepts have provided stimulating analogies and have helped to link studies as apparently unrelated as thermodynamics and semantics. It is evidently a branch of applied mathematics of wide general interest.

The purpose of this paper is to present the basic ideas of the theory in elementary terms; the application of these ideas to practical problems usually requires a technical knowledge of the particular field of application. Though the theory could be given concisely in abstract terms, it is helpful in an introductory exposition to use one of the more concrete terminologies in which it can be expressed: the terminology of its founders will be used.

"*Information*"

In telecommunication engineering, "information" can be defined as any signal or sequence of signals the engineer may be expected to store or transmit. Professionally he is not interested in the meaning, interpretation, or truth of the signals (these concern only the sender and receiver) but solely with the technical problem of transmitting them within specified limits of loss, distortion or delay. Information is usually transmitted by first encoding it into a convenient form, e.g. by making marks on paper, by punching holes in cards, by converting it into patterns of electrical impulses or of electromagnetic waves. At the receiving end of the transmission the engineer has to provide a decoder to restore the received signals to their original form. In practical communication channels there is usually a risk of loss or distortion of the transmitted signals; any distortion which is systematic can, at least in principle, be corrected in the receiver, but any distortion which is random in its effects and in its occurrence is called "noise". The most interesting results of the theory concern the transmission of continuous signals through noisy channels, but we first consider noiseless channels transmitting discrete signals in order to define some fundamental measures.

The original exposition of the theory by C. E. Shannon (1948) is based on a consideration of long random sequences. His method is abstract, elegant and of wide generality, but P. M. Woodward (1953) and S. Goldman (1953) have shown that a less sophisticated approach, in which single independent message signals are first considered, is intuitively more readily acceptable. Their method of approach is therefore more suitable for an introductory exposition and will be adopted here though, unfortunately, satisfactory proofs of the fundamental theorems of Information Theory (which will not be attempted here) depend on the properties of ergodic stochastic processes.

To avoid conflating the technical and everyday senses of the word " information " the symbol " I " will be used (except in sub-headings and in " Information Theory ") wherever the word is used in its defined technical sense.

The Measure of Information Capacity

The I-capacity of the physical system in which a signal is stored, i.e. a piece of paper, a card, a magnetic tape, etc., depends on the number of distinguishable states or " complexions " that the system can have. For example, a wheel of a decimal digital computing machine can store any one of 10 digits in its 10 possible positions, i.e. it has 10 complexions. Two such wheels in combination can store any one of 100 numbers and so, together, they have 100 complexions. More generally, r similar storage units each with n complexions would have, in combination, n^r complexions. The first step in the development of the theory was taken by R. V. L. Hartley (1928) when he defined the I-capacity of a store as $\log N$, where N is its number of complexions. Thus, by Hartley's definition, the I-capacity of a store with r similar units is r times as great as the capacity of the unit store ; five punch-cards thus have five times the capacity of one such card. The definition thus conforms with our intuitive ideas of storage capacity.

The " Bit " of Information

The base of the logarithm in Hartley's measure is arbitrary ; any base that is convenient may be used to establish the unit. But since the simplest possible I-store has two complexions—" yes-no ", " in-out ", " 0 - 1 ", etc.—such a store is convenient as the basic unit. The name of this unit, when 2 is taken as the base of the logarithm, was originally the " binary digit ", but this name was quickly and inevitably condensed to the " bit " (though the term " binit " is also used). Thus a device able to store any 20-digit decimal number must have 10^{20} complexions, and its I-capacity is 20 " decimal digits " or $20 \log_2 10 = 66{\cdot}4$ bits.

The Measure of Information Content

The next step was to establish a measure of the I-content of a signal—a step which was taken (in more general terms than will be given here) by Shannon (1948). Consider the set of all possible signals whose *a priori* probabilities of being sent are known to be $p_1, p_2, \dots p_i, \dots p_n$, where $\Sigma p_i = 1$. We do not know precisely which of these signals will be sent but we need a measure which will enable the engineer to compare the average I-content of the signals he is to transmit with the I-capacity of the channel or store he has available for them. Shannon suggested that the measure required, a function of the p_i, should have the following properties :

(*a*) It should be continuous in the p_i.

(*b*) For cases in which all the p_i are equal it should be a monotonic increasing function of n.

(*c*) It should be additive in the sense that the measure applied to two independent successive signals should yield the same result as the sum of the results obtained by applying the measure to the two signals separately. For example, if the set of signals is A and B with *a priori* probabilities p_1 and p_2 respectively ($p_1 + p_2 = 1$), then the set of two successive signals will consist of the four signals AA, AB, BA, BB with *a priori* probabilities p_1^2, p_1p_2, p_2p_1 and p_2^2 respectively. The required function H must then satisfy the relation

$$2H(p_1, p_2) = H(p_1^2, p_1p_2, p_2p_1, p_2^2).$$

Shannon formally proved that such a function exists and that its required properties are satisfied by $-k\Sigma p_i \log p_i$ where k is a constant, as can easily be verified for the simple example just given. The value of k and the base of

the logarithms determine the unit of the measure. If we put $k=1$ and take the logarithmic base as 2, the unit is again the bit.

This function H, the measure of I-content, has interesting properties and can be interpreted in several ways. First consider the case in which only two signals are possible, with probabilities p and $1-p$. For this set

$$H=-p\log p-(1-p)\log(1-p).$$

If H is regarded as a function of the continuous variable p, which, as a probability, ranges from 0 to 1, we see that H is zero when $p=0$ or 1, and that it reaches a maximum when $p=\frac{1}{2}$, for which $H=-\frac{1}{2}\log_2\frac{1}{2}-\frac{1}{2}\log_2\frac{1}{2}=1$ bit.

The function H is thus a measure of the prior uncertainty or ignorance of the signal to be transmitted. If $p=0$ or 1, then there is no uncertainty about the signal and its I-content is nil; otherwise there is always some uncertainty, and this uncertainty is greatest when the two possible signals are equiprobable. If there are more than two signals, say n, then again H is zero if any $p_i=1$ and has a maximum of $\log_2 n$ bits when the n signals are equiprobable.

It follows that in any enquiry in which the possible answers are known to be finite and discrete, the questions yield the most information *on average* if they are framed to equalize, or to equalize as nearly as possible, the probabilities of the possible answers. If the purpose of the enquiry is to identify one among 2^n apparently similar objects, then random questioning yielding "Yes" or "No" answers would need an average of 2^{n-1} questions; whereas systematic application of the rule, i.e. making the answers "Yes" and "No" equiprobable, would identify the object in only $\log_2 2^n$ or n questions (since each question for $p=\frac{1}{2}$ would elicit 1 bit and n bits are needed for the identification). The rule is applied in the radio game of Twenty Questions, e.g. "Is the article worn above or below the belt?", and G. A. Barnard (1951) has shown how to apply it to the well-known problem of identifying one defective coin among twelve similar coins in three weighings. The rule has of course more serious applications.

Since for a given set of signals $\Sigma p_i=1$, the function H can be regarded as $(-\Sigma p_i\log p_i)/(\Sigma p_i)$, i.e. as the "expectation" of $-\log p_i$, or as the arithmetic mean of the terms $-\log p_i$ weighted according to their probabilities. From this point of view a single signal of *a priori* probability p can be interpreted as having an I-content of $-\log p$, so that the smaller the probability of the signal the greater is its I-content. This conforms with our everyday attitude to information in one respect—the greater the unexpectedness of any news we hear the greater is its "news value". But this interpretation suggests that I-content can be measured in an absolute sense whereas it can be argued that only the difference of content between one I-state and another can be measured. Before a signal about some event E is received there is a state or level of uncertainty about E as one among many other possible events. When the signal is received the level of uncertainty is changed, but only with respect to E. And so, if the initial level of ignorance $I(i)$ can be measured by $-\log p+I$ where p is the *a priori* probability of the signal about E, and I is the measure of all other uncertainties, then the level of ignorance after receiving the signal, $I(f)$, is $-\log p'+I$, where $-\log p'$ measures any remaining uncertainty about E. So that

$$\begin{aligned}\text{I-gain}&=I(i)-I(f)\\&=(-\log p+I)-(-\log p'+I)\\&=-\log(p/p')\qquad\qquad(1)\end{aligned}$$

If the signal is reliable then $p'=1$ and the I-gain is $-\log p$ as before. As Woodward points out, the function H is thus a measure of differences of uncertainty rather than an absolute measure of I-content, though it is in the latter sense that H is commonly used in the theory.

The Entropy of an Information System

The function $-k\Sigma p_i \log p_i$ is already well-known to physicists as the function derived by Boltzmann in statistical mechanics. He called this function the *entropy* of the system in which p_i is the probability of a given particle being in cell i of its phase-space. By analogy with Boltzmann's function, Shannon called his function H the entropy of the set of probabilities p_i and thus suggested some interesting analogies between thermodynamic and information systems. Shannon's entropy function H, used as a measure of average I-content, or, more correctly, of differences between levels of uncertainty, is a basic concept in the development of Information Theory.

The Coding of Signals

One of the well-known codes used in telecommunications is the Morse code, which represents the letters of the alphabet as combinations of the dot and the dash, and which can be transmitted, for example, as flashes of light or as pulses in an electrical circuit. In the human nervous system all the signals the central nervous system receives from the external world, through all the senses, are transmitted along the appropriate nerve channels as electrical impulses. These impulses are all of the same kind and are of equal magnitude in any one channel, but the frequency of the impulses varies, within certain limits, approximately as the logarithm of the intensity of the stimulus (Fechner's Law). Language is also a code and its properties have been investigated by Shannon and others from the point of view of Information Theory.

The structure of the Morse code illustrates a principle of some importance in telecommunications; it is that the more probable signals should have the simpler code symbols for economy in transmission. Thus in the Morse code the letter e, the most frequent letter in English words, is coded as a single dot, while the letter z, which occurs comparatively rarely, is represented by the combination " dash dash dot dot ". (Note that the Morse code is not binary —it needs " spaces " as well as dots and dashes.) General methods of devising optimum codes for signal systems have been established by R. M. Fano, D. A. Huffman and others. For example, if the eight signals A, B, C, D, E, F, G and H are equiprobable, then an optimum binary code would be given by Code I.

Code I

A	000	E	100
B	001	F	101
C	010	G	110
D	011	H	111

But if the eight signals have *a priori* probabilities proportional to 8, 2, 1, 1, 1, 1, 1, and 1 respectively, then Huffman's method yields Code II as an optimum binary code.

Code II

A	0	E	1000
B	110	F	1001
C	1010	G	1110
D	1011	H	1111

This code clearly conforms with the principle that the more probable signals be given the simpler combinations of the code symbols. Moreover, with such a code any sequence of binary digits can be decoded without ambiguity and so no " spaces " are needed—it is truly binary.

The entropy of the equiprobable signals of Code I is $\log_2 8$ or 3 bits/signal

(the maximum attainable with 8 signals) or 1 bit/symbol. The entropy of the signals of Code II is, however, given by :

$$H = -\tfrac{1}{2}\log\tfrac{1}{2} - \tfrac{1}{8}\log\tfrac{1}{8} - 6\,.\,\tfrac{1}{16}\log\tfrac{1}{16}$$
$$= 2\tfrac{3}{8} \text{ bits/signal.}$$

Since the average length of the signals of Code II is $2\frac{3}{8}$ symbols, the entropy of the symbols is, as for Code I, 1 bit/symbol, indicating that an optimum coding has been found with equal frequencies of 0's and 1's. It may be noted that in both codes the entropy per signal measured in bits is equal to the mean signal length in binary symbols. This equality can be shown to be generally true for optimum coding but it is not always possible to achieve the optimum coding exactly.

Redundancy and Compression

Using the same signals the fraction of the maximum entropy attained by a code is called its *relative entropy*, and the difference between unity and this fraction is called the *redundancy* of the code. The relative entropy of the signals of Code II is $2\frac{3}{8}/3$ or 19/24 and its redundancy is therefore 5/24 or 20·1%. Redundant signal systems can be compressed into non-redundant systems using the same signals or alphabet. For example, the following sequence of 24 signals was selected at random (using random numbers) from a signal system with the probability distribution of Code II :

AAAE FGDA GBAA AAEG BACA HAAF ...

(They are grouped in fours for ease of reading.) Coded into the binary digits of Code II the sequence becomes :

0001000 1001111010110 1110110000 001000110 110010100 1111001001

which, ignoring the last digit, can be decoded by means of Code I to give the 19 signals :

AECD GFFG GACB GGCE HEE ...

And so the longer message of 24 signals can be compressed (almost exactly) into the shorter sequence of 19 similar signals, each sequence containing 57 bits ($24 \times 2\frac{3}{8}$ or 19×3).

Shannon has shown that the redundancy of the English language is about 50%, i.e. it is possible to restore an English text when about half the letters are deleted at random, partly because successive letters are not wholly independent of those that precede them (e.g. a *q* is almost always followed by a *u*). The fact that English is redundant is not wholly disadvantageous to its users ; for example, it permits intelligible discourse in the presence of noise, or the reading of badly written manuscripts, and, moreover, its 50% redundancy makes possible the construction of large two-dimensional cross-word puzzles!

Channel Capacity

In most communication systems the channel capacity made available to the user is very much greater than he strictly needs. For example, as Wiener has pointed out, the telephone subscriber connected to an automatic exchange can " ring up " with equal facility any other subscriber on the exchange whether the probability of his requiring that particular connection is great or small ; he can speak over the telephone in any language he commands though in Britain the English language is in predominant use. From the point of view of Information Theory the telephone system therefore provides an extravagantly large channel capacity, though from economic and engineering points of view it would now be costly and complicated to restrict its channel capacity to the level that Information Theory indicates to be the optimum. In radio, radar, and television, however, engineers are faced with problems of

designing transmitters and receivers (i.e. coders and decoders) which will make more effective use of given channels, e.g. frequency bandwidths, and so make more channels available and reduce interference between them.

The theory of the discrete noiseless channel is summarized in a fundamental theorem. Shannon has formally shown that if the entropy of a signal system is H bits/symbol and the capacity of a channel is C bits/second, then it is possible to transmit up to a rate of $\frac{C}{H} - \epsilon$ symbols/second, where ϵ is arbitrarily small, by optimum coding of the signals. If the signals are generated by a source which feeds into a channel, then H is the measure which determines the channel capacity to transmit the signals without delay. The most important aspects of this theorem (which will not be proved here) are the originality of its approach and its demonstration of the applicability to practical communication channels of the concepts of I-capacity and I-entropy.

The Model of the Noisy Channel

The simple theoretical model of the discrete channel is thus a stochastic process in which the signals transmitted are independent selections from a finite set of signals whose *a priori* probabilities are known. For such a model, and, as Shannon showed, for the more general case (as in language) where there is some inter-signal influence, the measure of the average I-transmission is the entropy of the probability distribution of the signals. When " noise " is introduced into the model a second stochastic process occurs : some of the signals transmitted are incorrectly received as other signals, not necessarily of the input set. A simple model of the noisy channel can be described by the set of transition probabilities of receiving signal y if signal x is sent, for all x and y, assuming that the noise affects successive signals independently ; these transition probabilities then specify the noise characteristics of the channel. In the noisy channel the measure of the I-output is no longer the entropy of the I-input, since some of the input is changed by the noise, nor can it be the entropy of the I-output, since some of the received signals are known to be false though the false signals cannot be specified. We have therefore to examine the other entropies that can now be computed, explore the relations between them, and establish a measure of the I-transmission of the noisy channel compatible with the results obtained for the noiseless channel.

The Probability Distributions of a Noisy Channel

Consider a simple model of a noisy channel in which the input and the output consist of the same finite set of signals. The simplest model of this kind is a channel transmitting the binary digits 0 and 1 and whose characteristics are specified in Table I. The *transition probabilities*, $p_x(y)$, describe the noise characteristics. For example, the probability of output 1 with input 0 is $q_0 (p_0 + q_0 = 1)$. Let p and q (where $p + q = 1$) be the probabilities of

TABLE I

Input	x	0		1	
Output	y	0	1	1	0
Transition	$p_x(y)$	p_0	q_0	p_1	q_1
Input	$p(x)$	p		q	
Joint	$p(x, y)$	pp_0	pq_0	qp_1	qq_1
Output	$p(y)$	$(pp_0 + qq_1)$ for $y = 0$		$(qp_1 + pq_0)$ for $y = 1$	
Conditional	$p_y(x)$	$\frac{pp_0}{pp_0 + qq_1}$	$\frac{pq_0}{qp_1 + pq_0}$	$\frac{qp_1}{qp_1 + pq_0}$	$\frac{qq_1}{pp_0 + qq_1}$

the inputs of 0 and 1 respectively. Then the *joint probabilities* $p(x, y)$ are the probabilities of x (i.e. 0 or 1) being the input and of y (also 0 or 1) being the output; thus the joint probability of input 0 and output 0 is pp_0. The total probability of output 0 from all input signals, i.e. the *output probability* $p(y)$ when y is the digit 0, is $(pp_0 + qq_1)$. The *conditional probability*, $p_y(x)$, is the probability that x was the input if y is the output.

Evidently for such a model all the following sums are equal to unity: $\sum_x p(x)$ and $\sum_y p_x(y)$ for any given x; $\sum_x p_y(x)$ for any given y; $\sum_y p(y)$; and $\sum_x \sum_y p(x, y)$.

If on average the signal x is sent n_x times in n signals, and the signal y is sent n_y times in n signals, then $p(x) = n_x/n$ and $p(y) = n_y/n$. If, of the n_x signals x, n_{xy} are received as y, then $p(x, y) = n_{xy}/n$. Hence:

$$p(x, y) = \frac{n_{xy}}{n} = \frac{n_x}{n} \cdot \frac{n_{xy}}{n_x} = \frac{n_y}{n} \cdot \frac{n_{xy}}{n_y}$$

giving

$$p(x, y) = p(x)p_x(y) = p(y)p_y(x) \qquad \text{.........................(2)}$$

There are thus five related probability distributions and their associated entropies to consider in the model of the noisy channel.

The Measure of the Information transmitted by a Noisy Channel

Shannon justifies the following expression as a measure of the average I-transmission of a noisy channel:

$$I = H(x) + H(y) - H(x, y) \qquad \text{.........................(3)}$$

where $H(x) = -\Sigma p(x) \log p(x)$, the entropy of the input.

$H(y) = -\Sigma p(y) \log p(y)$, the entropy of the output.

and $H(x, y) = -\Sigma\Sigma p(x, y) \log p(x, y)$, the joint entropy.

Let us consider this expression. If the channel is free from noise, then for all x and y, $p(x) = p(y)$ and $p(x, y) = p(x)$ for all cases in which x and y are the same signal and is zero otherwise. Hence, for this noise-free channel $H(x) = H(y) = H(x, y)$ and expression (3) reduces to $I = H(x)$ and thus conforms with the result already established.

If x and y are independent, then $p(x, y) = p(x)\,p(y)$ for all x and y and so

$$\begin{aligned} H(x, y) &= -\sum_x \sum_y p(x, y) \log p(x, y) \\ &= -\sum_x \sum_y p(x)\,p(y)\{\log p(x) + \log p(y)\} \\ &= -\sum_x p(x) \log p(x) - \sum_y p(y) \log p(y) \\ &= H(x) + H(y). \end{aligned}$$

In this case expression (3) is zero, implying that the I-transmission is nil, as we should expect when the signals received are independent of the signals sent. For example, if $p(0, 0) = p(0, 1) = p(1, 0) = p(1, 1) = \frac{1}{4}$ are the joint probabilities of a channel transmitting the binary digits 0 and 1, then for the input, $p(0) = p(1) = \frac{1}{2}$ and similarly for the output. A sample of the transmission of such a channel is:

x	0011	0000	0111	1100 ...
y	1001	0001	0101	1011 ...

This was obtained by writing 0 for the even numbers and 1 for the odd numbers in two lines of a table of random numbers. The double sequence satisfies the condition $p(x, y) = p(x)\,p(y)$, and though in this sample 9 out of 16 symbols are correctly " transmitted ", we have

$$I = H(x) + H(y) - H(x, y) = 1 + 1 - 2 = 0.$$

As the expectation of "correct" symbols in a similar long sequence of $2n$ signals is n, any random sequence of 0's and 1's with $p(0)=p(1)=\frac{1}{2}$, such as might be obtained by tossing a penny and putting 0's for tails and 1's for heads, would on average give as many correct signals as any sequence actually transmitted.

If $p(0, 0)=p(1, 1)=\frac{3}{8}$ and $p(0, 1)=p(1, 0)=\frac{1}{8}$ it will be found that half the I-content is lost in transmission although three out of four digits are correct on average. If $p(0, 0)=p(1, 1)=\frac{1}{8}$ and $p(0, 1)=p(1, 0)=\frac{3}{8}$, then the symbols are more often incorrect than correct, but by systematically correcting the signals, reading 1's as 0's and 0's as 1's, half the I-content is transmitted as for the previous case.

As a numerical example of the noisy channel consider the data of Table II. The channel transmits the signals a, b, c and d and its noise characteristics are specified by the given value of the transition probabilities $p_x(y)$. Given that $p(x)=\frac{1}{4}$ for all four signals, the values of $p(x, y)$ and of $p_y(x)$ can be written down. For the output signals y it will be seen that

$$p(a)=p(b)=\tfrac{3}{16} \quad \text{and} \quad p(c)=p(d)=\tfrac{5}{16}.$$

TABLE II

x	a	a	a	a	b	b	b	c	c	d
y	a	b	c	d	a	b	c	a	c	d
$p_x(y)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{4}$	1
$p(x, y)$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{3}{16}$	$\frac{1}{4}$
$p_y(x)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{5}$	$\frac{1}{3}$	$\frac{3}{5}$	$\frac{4}{5}$

Computing the entropies we have :

$$H(x)= -4\,.\,\tfrac{1}{4}\log\tfrac{1}{4} \qquad =2 \text{ bits/signal}$$

$$H(y)= -2\,.\,\tfrac{3}{16}\log\tfrac{3}{16}-2\,.\,\tfrac{5}{16}\log\tfrac{5}{16}=1{\cdot}98 \text{ bits/signal}$$

and

$$H(x, y)= -7\,.\,\tfrac{1}{16}\log\tfrac{1}{16}-\tfrac{1}{8}\log\tfrac{1}{8}-\tfrac{3}{16}\log\tfrac{3}{16}-\tfrac{1}{4}\log\tfrac{1}{4} =3{\cdot}09 \text{ bits/signal}$$

Hence

$$I=2+1{\cdot}98-3{\cdot}09 \qquad =0{\cdot}89 \text{ bits/signal.}$$

A typical transmission of such a channel (derived by the use of a table of random numbers) is :

x	$a\ b\ a\ a$	$d\ b\ d\ c$	$c\ b\ c\ c$	$c\ c\ d\ d$	$a\ a\ a\ b\ldots$
y	$c\ b\ b\ a$	$d\ b\ d\ c$	$c\ b\ c\ c$	$c\ a\ d\ d$	$b\ d\ a\ c\ldots$

The Equivocation of a Noisy Channel

Expression (3), rewritten as $H(x)-\{H(x, y)-H(y)\}$ can be interpreted as the difference between $H(x)$, the average I-input, and $\{H(x, y)-H(y)\}$, which is a measure of the average uncertainty caused by the noise. Shannon called this uncertainty the *equivocation* of the channel and showed that it can be more concisely expressed. Using the identities (2) we have :

$$\begin{aligned} H(x, y) &= -\sum_x\sum_y p(x, y)\log p(x, y) \\ &= -\sum_x\sum_y p(x)\,p_x(y)\{\log p(x)+\log p_x(y)\} \\ &= -\sum_x p(x)\log p(x)-\sum_x\sum_y p(x)\,p_x(y)\log p_x(y) \\ &= H(x)-\sum_x\{p(x)\sum_y p_x(y)\log p_x(y)\}. \end{aligned}$$

The quantity $-\sum_y p_x(y)\log p_x(y)$ is the output entropy for a given input

signal x; the expectation of this entropy for all inputs is the mean of all such quantities weighted according to $p(x)$. This is called the *conditional entropy* of y and is denoted by $H_x(y)$. So that finally,

$$\begin{aligned} H(x, y) &= H(x) + H_x(y), \\ &= H(y) + H_y(x) \end{aligned}$$

similarly, where $$H_y(x) = \sum_y p(y) \sum_x p_y(x) \log p_y(x) \qquad \text{.........................(4)}$$

Hence we can write:

$$\begin{aligned} I &= H(x) - \{H(x, y) - H(y)\} \\ &= H(x) - H_y(x) \\ &= H(y) - H_x(y) \qquad \text{.........................(5)} \end{aligned}$$

where $H_y(x)$ is the equivocation.

For the data of Table II the output entropies are 1·51, 0·84, 1·42, and 0·72 bits/signal for the signals a, b, c and d respectively; and their weighted mean, $H_y(x)$, is 1·11 bits/signal, giving $I = H(x) - H_y(x) = 0{\cdot}89$ bits/signal as already found.

The use of the expression (5) as a measure of the I-transmission can be justified by considering the noisy channel from the point of view of an observer who is able to compare the input and output signal sequences. If he were provided with an auxiliary noise-free channel to the receiver he could correct the false signals as they occurred. In principle all he need do is to send some sequence of signals such as "$- - a - - c\ b - -$" where a dash denotes that the output signal is correct and the letters give the correct signals to be substituted for those received. However, to send a signal saying, in effect, that "The signal s that has just been received has been correctly received as s" is equivalent to transmitting the signal s itself down the auxiliary channel. The only signals over which he need take no action are those which are never received in error for another signal, for which $p_y(x) = 1$ and $-\Sigma p_y(x) \log p_y(x) = 0$. All ambiguous signals received will have to be either confirmed or corrected by the observer. The measure of the average I-input of the observer is therefore the mean of the terms $-\sum_x p_y(x) \log p_y(x)$ for each signal y, weighted by $p(y)$; and this is $H_y(x)$, the equivocation. As this is the I-input that must be sent down the auxiliary noiseless channel to correct the signals actually received, $H(x) - H_y(x)$ is the appropriate measure for the I-transmission of the noisy channel.

The Capacity of a Noisy Channel

As some of the I-input of a noisy channel is destroyed it might appear that, though the probability of error could be reduced by repeating the messages or by using redundant codes, the channel could never be used with complete certainty and that reduction of the equivocation could be won only at the cost of reducing the rate of transmission. But perhaps the most surprising result of Shannon's theory is his proof that it is possible to send information at a definite rate through the channel with as little equivocation as may be desired, but that this can be done only by matching the input signals to the noise characteristics of the channel, i.e. by suitable encoding.

Shannon first defines the I-capacity of a noisy channel as $C = \text{Max } I$ where I is given by the expression (5). In maximising any of these expressions we take the transition probabilities $p_x(y)$ (the noise characteristics) as fixed and vary the input probabilities $p(x)$. Shannon then shows that if the I-input $H(x)$ is not greater than C, then, with suitable coding, it can be transmitted with equivocation ϵ, where ϵ is arbitrarily small; if $H(x) > C$, then the equivocation is at least $H(x) - C$. Shannon's proof (which will not be given here)

is however an existence proof and does not indicate how optimum codes should be devised.

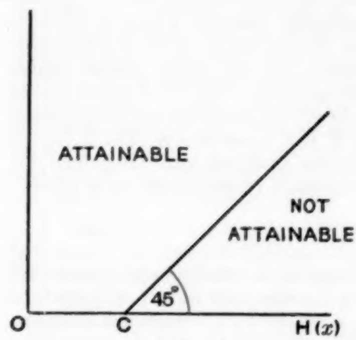

Fig. 1.

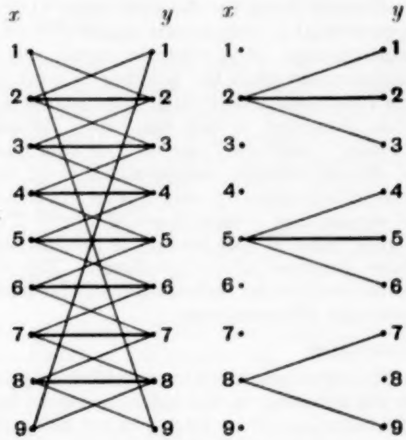

Fig. 2.

Yet the ability of a noisy channel to transmit without equivocation can be demonstrated in simple cases. For example, the characteristics of the channel illustrated in the figure permit easy recognition of the optimum code. This channel transmits the digits 1, 2, 8 ... 9, with equiprobable transitions as shown in the figure ; thus, if the digit 3 is transmitted it is equally likely to be received as 2, 3 or 4.

Consider $C = \max I = \max \{H(y) - H_x(y)\}$.

For each digit, $-\Sigma p_y(x) \log p_y(x) = -3 \,.\, \frac{1}{3} \log \frac{1}{3} = \log 3$

and therefore, $H_x(y) = \sum_x p(x) \log 3 = \log 3$.

Hence $H_x(y)$ is here constant for all distributions of $p(x)$. The entropy of the I-output, $H(y)$, is a maximum when all the $p(y)$ are equiprobable, and this condition must therefore be met to achieve maximum I-capacity. We then have

$$H(y) = -\Sigma p(y) \log p(y) = -9 \,.\, \tfrac{1}{9} \log \tfrac{1}{9} = \log 9$$

Hence $C = \max I = \log 9 - \log 3 = \log 3$.

This I-capacity can be achieved by selecting 2, 5 and 8 (for example) as the only digits to be transmitted and coding the digits thus :

Digit	1	2	3	4	5	6	7	8	9
Code	22	25	28	52	55	58	82	85	88

The condition that all $p(y)$ must be equiprobable will be met if the three transmitted digits 2, 5, 8 are also equiprobable. In this case we then have

$$H(x) = -3\,\tfrac{1}{3}.\ \log \tfrac{1}{3} = \log 3 \text{ as required.}$$

With this input and the given coding the maximum I-transmission occurs with complete reliability since all the received signals can be decoded without ambiguity. The channel thus transmits to its optimum I-capacity without equivocation, but at the cost of needing two digits for each single digit of the original message.

Many noise-defeating codes can be found in the literature of Information Theory, but they are ingenious *ad hoc* solutions of particular problems rather than illustrations of general methods.

The Continuous Channel

The mathematician might expect the theory of the continuous channel to be derived from the discrete channel theory by a limiting process. It might appear that a continuous signal $f(t)$ of finite length could be regarded as a limiting case of a discrete signal sequence, but such an approach would suggest that since the number of complexions of such a signal is unbounded, the I-content of any finite continuous signal could be increased without limit. This, however, is not the case: in any practical channel there is always "noise", and the "power" of this noise can be shown to limit the number of distinguishable complexions to $(1+r)$ where r is the ratio (mean signal power/mean noise power). But even in the theoretical absence of noise it can be shown that a wave-form of length T seconds which contains no frequencies higher than W per second is determined by $2\ WT$ parameters, i.e. it has a finite number of complexions. The continuous channel can therefore be considered as an elaboration of the discrete channel rather than as a fundamentally different case.

Conclusion

Though the telecommunications engineer may professionally be uninterested in the meaning of the information he handles, no one else is. It is therefore not surprising that attempts are being made, notably by the logician Carnap, to extend the theory to include the semantic as well as the I-content of signals. Attempts are also being made to develop from it a theory of decision applicable to the study of social behaviour. It is perhaps in the stimulus it provides for those seeking measures of hitherto unmeasurable processes that the main value of Information Theory lies.

B. C. B.

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1861. *Fractional Drunkenness.*

Four cases of drunkenness have been reported during the past year: not serious you would say, but it represents a 50 per cent. increase.—"Country Sessions", in *The Countryman*, Summer 1955.

A. P. R.

1862. *The Crooked Spire of Chesterfield.*

Tipsily crowning every central vista, like a tall paper hat at a late stage in a Christmas party, rises the bent and twisted spire of the parish church, nearly eight feet out of true in one direction and more than three feet in two others.—*Manchester Guardian*, Sept. 9 1955.

A. P. R.

ON THE STEREOGRAPHIC PROJECTION OF THE SPHERE

By A. P. Stone

1. The stereographic projection of the sphere is a perspective projection from a point on the surface onto the opposite diametral plane. Analytical geometry provides a unified method of treating the stereographic projection and problems in spherical trigonometry. The basic result is the equation for the projection of a general circle (eq. 1). This is applied to the polar, equatorial and oblique stereographic projections, and a simple proof of Cayley's theorem [1] is obtained. Next, the cosine formula for a right-angled spherical triangle and the relation between angles measured along different great circles are derived. Finally, the construction used on astrolabes for determining planetary time [2] is shown to be the stereographic projection of a problem on the sphere, which is solved.

2. Any circle on the sphere may be specified as lying in a plane (l, m, n) whose perpendicular distance from the centre of the sphere is p. Taking the point of projection as origin and the centre of the sphere (radius a) at the point $(0, 0, a)$, the cone through the point of projection and the given circle is

$$x^2+y^2+z^2-2az(lx+my+nz)/(p+an)=0$$

and the projection of this circle on the plane $z=a$ is

$$(p+an)(x^2+y^2)-2a^2(lx+my)+a^2(p-an)=0, \quad \dots\dots\dots\dots\dots(1)$$

a circle of radius $a\sqrt{a^2-p^2}/(p+an)$ with its centre at the point $\dfrac{a^2}{p+an}(l, m)$.

3. In the polar stereographic projection, the point of projection is the South pole. A parallel of latitude, colatitude ζ, is given by $(l, m, n)=(0, 0, 1)$, $p=a\cos\zeta$. Substituting in (1), its projection is $x^2+y^2=a^2\tan^2\frac{1}{2}\zeta$. The radius gives the well-known formula for a stereographic scale, $r=a\tan\frac{1}{2}\zeta$. For a meridian, $n=p=0$ and the projection is a straight line through the origin, $lx+my=0$.

In the equatorial projection, the point of projection is taken on the equator. The meridians and parallels project into two orthogonal systems of coaxial circles.

Meridians : $x^2+y^2-2amy/n-a^2=0$.

Parallels : $x^2+y^2-2ax\sec\zeta+a^2=0$, of radius $a\tan\zeta$.

4. The isogonal property of the stereographic projection may easily be demonstrated in the particular case of two orthogonal circles, one of which is a great circle. The projection of any great circle cutting the general circle orthogonally is

$$n'(x^2+y^2)-2a(l'x+m'y)-a^2n'=0 \quad \dots\dots\dots\dots\dots\dots\dots(2)$$

where $ll'+mm'+nn'=0$, while the condition for (1) and (2) to be orthogonal is

$$(ll'+mm'+nn')/n'(p+an)=0.$$

If n' or $p+an$ vanishes, the corresponding circle projects into a straight line and the condition for orthogonality is that this line should pass through the centre of the other circle. For $n'=0$, the point $\dfrac{a^2}{p+an}(l, m)$ must lie on $l'x+m'y=0$. For $p+an=0$, $\dfrac{a}{n'}(l', m')$ must lie on the line $lx+my+an=0$. Both cases require $ll'+mm'+nn'=0$.

5. In the oblique stereographic projection, the plane of projection is the horizon at a point of colatitude u (c.f. fig. 1). The direction cosines for the

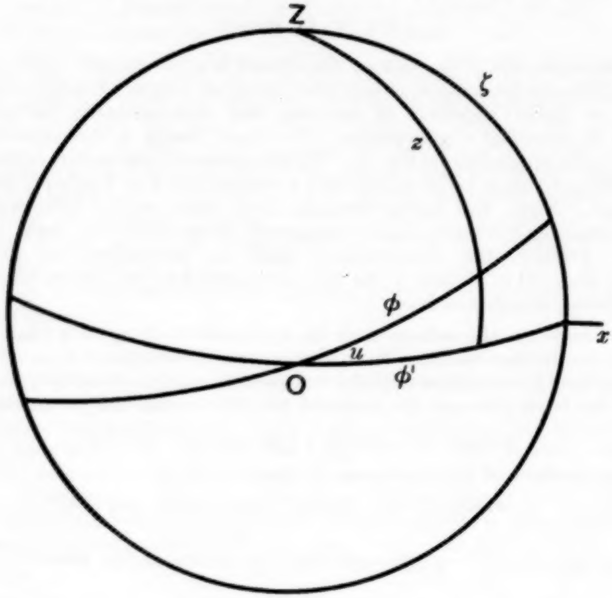

FIG. 1.

meridians have the form $(l, m, n) = (\cos u \cos \phi, \sin \phi, \sin u \cos \phi)$ and their projections are

$$x^2 + y^2 - 2a(x \cot u + y \operatorname{cosec} u \tan \phi) - a^2 = 0, \qquad \text{................(3)}$$

a coaxial system whose centres lie on the line $x = a \cot u$. The radius of a circle is $a \operatorname{cosec} u \sec \phi$ and the smallest circle is of radius $a \operatorname{cosec} u$.

The parallel of colatitude ζ' is given by $(l, m, n) = (-\sin u, 0, \cos u)$, $p = a \cos \zeta'$ and projects into

$$(x^2 + y^2)(\cos \zeta' + \cos u) + 2ax \sin u + a^2(\cos \zeta' - \cos u) = 0.$$

The radius of this circle is

$$\begin{aligned} r &= a \sin \zeta' / (\cos \zeta' + \cos u) \\ &= 2a \operatorname{cosec} u \tan \tfrac{1}{2}\zeta' \tan \tfrac{1}{2}u / (1 - \tan^2 \tfrac{1}{2}\zeta' \tan^2 \tfrac{1}{2}u) \qquad \text{............(4).} \end{aligned}$$

The meridians and parallels project into orthogonal coaxial systems, exactly as in the equatorial projection for a sphere of radius $a \operatorname{cosec} u$ except that the circles representing parallels refer to different latitudes in the two cases. Since the appropriate equatorial projection of a parallel of colatitude ζ is of radius $a \operatorname{cosec} u \tan \zeta$, the relation between the colatitudes is, by (4),

$$\tan \tfrac{1}{2}\zeta = \tan \tfrac{1}{2}\zeta' \tan \tfrac{1}{2}u.$$

This constitutes Cayley's theorem.

6. Two formulae of spherical trigonometry will now be derived. In the oblique stereographic projection, the equator projects into the circle

$$x^2 + y^2 + 2ax \tan u - a^2 = 0 \qquad \text{................................(5)}$$

The meridian at angle ϕ along the equator, measured from O (fig. 1), projects into (3), meeting (5) where $x+y\cos u\tan\phi=0$. Since the vertical circle through Z at azimuth ϕ' from O projects into $x+y\tan\phi'=0$, the angles are related by

$$\tan\phi'=\tan\phi\cos u \quad\text{.................................(6)}$$

The intersection of (3) and (5) is given by

$$x'^2(1+\sec^2 u\cot^2\phi)+2ax'\tan u-a^2=0$$

i.e.
$$x'=-a(\tan u\pm\sec u\operatorname{cosec}\phi).$$

The minus sign is the one appropriate to fig. 1. Then if $u+\zeta=\dfrac{\pi}{2}$,

$$\begin{aligned}\tan^2\tfrac{1}{2}z=(x'^2+y'^2)/a^2&=(\cot\zeta-\operatorname{cosec}\zeta\operatorname{cosec}\phi)^2/(1+\operatorname{cosec}^2\zeta\cot^2\phi)\\&=(1-\cos\zeta\sin\phi)^2/(1-\cos^2\zeta\sin^2\phi).\end{aligned}$$

Hence
$$\cos z=\cos\zeta\sin\phi. \quad\text{..(7)}$$

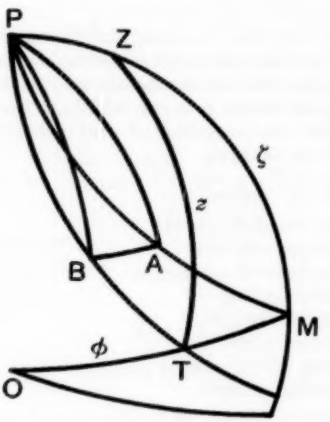

FIG. 2.

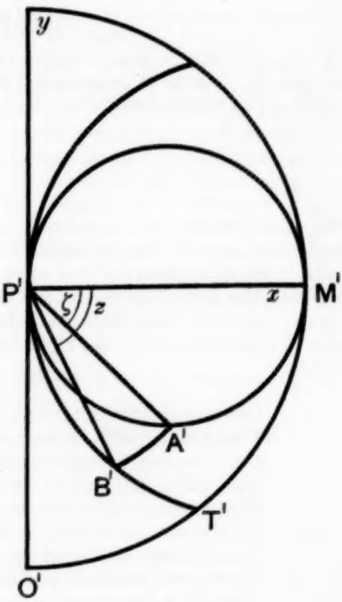

FIG. 3.

7. The hours of the day according to planetary time, or unequal hours of the day, are obtained by dividing the sun's diurnal arc into 12 equal parts. The construction used on astrolabes for finding planetary time is shown in fig. 3, which is the stereographic projection of fig. 2. It consists of 6 circles such as $P'T'$, referring to the 6 hours before and after noon. Planetary time from 0h. to 6h. is measured along $O'M'$, the projection of the equator; the sun's declination being neglected. Z is the zenith and ζ the sun's noon zenith distance. $P'A'$ is drawn at an angle ζ to $P'M'$, intersecting the circle $P'M'$ in A'. $P'B'=P'A'$ is then drawn at an angle z to $P'M'$ and T, the intersection of the circles $P'B'$ and $O'M'$, gives the planetary time when the sun's zenith

distance is z; directly if it is a.m. and subtracted from 12h. if it is p.m. The construction is made on the astrolabe with the help of the rotating sight-rule.

To give a proof of this method it is only necessary to show that if A in fig. 2 is determined by a procedure analogous to that for A', then AB, PT and the meridian PB meet in B. Since $p=an$ for PT, $P'T'$ has the equation $x^2+y^2=a^2x/p$ and since $O'M'$ is the circle $x^2+y^2=a^2$, T' is given by

$$p=x=a\sin\phi$$

if arc $OT=\phi$. The equation of $P'T'$ becomes $x^2+y^2=ax\operatorname{cosec}\phi$ and the diameter of this circle is $a\operatorname{cosec}\phi$. $P'M'$ has the equation $x^2+y^2=ax$ and meets $P'A'$ ($y=-x\tan\zeta$) in A': $a\cos\zeta(\cos\zeta, -\sin\zeta)$. Hence the equation of $A'B'$ is $x^2+y^2=a^2\cos^2\zeta$, which meets $P'T'$ where

$$x=a\cos^2\zeta\sin\phi=a\cos\zeta\cos z \text{ by (7)},$$

but this is also where $P'B'$ meets $A'B'$.

A.P.S.

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2. P. H. van Cittert, *Astrolabes*. E. J. Brill, Leyden, 1954.

1863. Complaints are frequently made that the education of the scientist or mathematician is too specialized, and that he would benefit from a curriculum which included more of the arts. To this end a new textbook is projected from which the following extract is taken and which it is confidently expected will push Godfrey and Siddons, Hall and Stevens, Durell and other such writers of conventional texts right out of the market.

Ad Classem Quartam

Contemplator item cum formas candida creta
Planitie in pulla descripserit atque viarum
Cunctos anfractus variarum expleverit—humum
Non aliter cultor mordaci vomere sulcat—
Illic conspicies, o terque quaterque beate,
(Si fors tu possis curvo dignoscere rectum)
Quadratam faciem et triquetram polygonaque pulchra.
Praecipue geminae coeunt perpendiculares
Lineae et amplexum triplum nova linea jungit.
Non tamen illis haec punctum concurrit in unum;
Trajicitur veluti pons quae ripas fluviales
Conjungat. Geminae quadratum quaeque venustum
Sustentant: neque item sine quadra tertia paret
Linea supposita. Praeclara figura profecto
Cantari digna est, hecatomben ac meruisse.
Nam si discideris quadrata minora gemella—
Nil magis admirandum audivi!—fragmina cuncta
Deinde reformari poterunt ut maxima quadra
Tota illis veletur ad unguem. Pythagorasque
Id primus verum vicit, vir ingeniosus. . . .

—[Per Mr. B. A. Swinden.]

THE BABYLONIAN QUADRATIC EQUATION

By A. E. Berriman

The purpose of this note is to show at a glance the significance of successive steps in the solutions to some of the quadratic equations that have come down in the cuneiform texts as examples of the mathematical instruction given to Babylonian students *c.* 1600 B.C. It is due to the translations made by O. Neugebauer in Germany and later, with A. Sachs, in the U.S.A. and to those published by Thureau-Dangin in France that this ancient material is available for general study, and, of course, it is from these sources * that the following examples are taken. The texts are rhetorical, they instruct the pupil to perform arithmetical operations with specific numbers derived from the problems ; in the translations these numbers retain their sexagesimal form † but here they are in our customary notation although expressed in a somewhat unusual way to facilitate, in particular, ready recognition of the re-entry of a number that has been temporarily left behind ; in the text the pupil is reminded that this is the number that " your head held ".

It is only by seeing every step that the validity of the sequence as a solution can be judged and the significance of the evidence as a whole be assessed. The examples given here are few out of many but I think they give a fair picture of the best available in the sources from which they are taken. Their primary importance lies in the use of standard procedures for their solution and, therefore, I have used symbols (in a parallel column) to show the meaning of each step in relation to the formula equivalent to the sequence as a whole. These symbols have been chosen so as to illustrate the Babylonian method as a prototype of later practice for in this respect the evidence has a special historical importance.

In the British Museum there is a tablet 13901 that originally contained twenty-four problems but some are damaged ; those that remain, however, are sufficient to reveal a carefully graduated course of instruction.

Here are the first and second problems on this tablet :

B.M. 13901, *No.* 1.

Text : " I added together the area and the side of my square. (3/4) ".

$$ax^2+bx=c\ ;\ a=1,\ b=1,\ c=(3/4)$$

Solution :	*Text*	*Formula*	
Write down	1	b	
	$(1/2)1=(1/2)$	$(b/2)$	
	$(1/2)^2=(1/4)$	$(b/2)^2$	
	$(1/4)+(3/4)=1$	$(b/2)^2+c$	$=Z$
	$=1^2$		$=(\sqrt{Z})^2$
	$1-(1/2)=(1/2)$	$\sqrt{Z}-(b/2)=x$	

* O. Neugebauer, *Mathematische Keilschrift-Texte* (1935).
F. Thureau-Dangin, *Textes Mathématiques Babyloniens* (1938).
O. Neugebauer and A. Sachs, *Mathematical Cuneiform Texts* (1945).

† Neugebauer uses a semi-colon to separate the whole number from the fraction, and commas to separate the other powers of 60. Thureau-Dangin's numbers are accented as in angular measure, the units being marked as degrees. For example :

$$1,\ 1,\ 1\ ;\ 1,\ 1=1''1'1^{\circ}1'1''=3600+60+1+(1/60)+(1/3600).$$

In the original cuneiform there is nothing to distinguish a fraction from its associated whole number, the figure sequence must be interpreted in the light of the context. Moreover, most of the problem texts belong to the Old-Babylonian period *c.* 1800 to 1600 B.C. when there was no sign for zero, and the scribe did not always indicate a numerical void by a blank space. In the Seleucid period beginning *c.* 300 B.C. (which covers the important astronomical texts) a zero was indicated by the period mark used to separate sentences.

B.M. 13901, *No.* 2.

Text : " I subtracted the side from the area of my square. 870 ".

$$ax^2 - bx = c \; ; \; \therefore \; ax^2 = bx + c \; ; \; a = 1, \; b = 1, \; c = 870$$

Solution :	*Text*	*Formula*
Write down	1	b
	$(1/2)1 = (1/2)$	$(b/2)$
	$(1/2)^2 = (1/4)$	$(b/2)^2$
	$(1/4) + 870 = 870{\cdot}25$	$(b/2)^2 + c \quad = Z$
	$= 29{\cdot}5^2$	$= (\surd Z)^2$
	$29{\cdot}5 + (1/2) = 30$	$\surd Z + (b/2) = x$

In the column of symbols, Z is an arbitrary simplification that I use for the square formed by $(b/2)^2 \pm ac$.

The above problems are prototype examples respectively of the categories that Al Khowarizmi (*c.* A.D. 820)* called :

Squares and roots = numbers	$ax^2 + bx = c$
Squares = roots and numbers	$ax^2 = bx + c$

There are also examples of his third category :

Squares and numbers = roots	$ax^2 + c = bx$

Al Khowarizmi taught his readers to reduce the coefficient of x^2 to unity.

" Where two squares or three or more or less be specified you reduce them to one single square, and in the same proportion you reduce the roots and simple numbers which are connected with it. For example, if 2 squares and 10 roots are equal to 48 dirhems then 1 square and 5 roots are equal to 24 dirhems."

Probably he was acquainted with the work of the Indian mathematician Brahmegupta who gave this rule (for the " elimination of the middle term ") in his course on astronomy (*c.* A.D. 620).†

" To the absolute number multiplied by four times the coefficient of the square, add the square of the coefficient of the middle term. The square root of the same, less the coefficient of the middle term, if divided by twice the coefficient of the square is the value of the middle term."

$$\text{If } ax^2 + bx = c \text{ then } x = (\sqrt{(4ac + b^2)} - b)/2a$$
$$= (\sqrt{(ac + (b/2)^2} - (b/2))/a$$

In the Babylonian texts the coefficient of x^2 was not eliminated until the final step when ax is multiplied by the reciprocal of a, this being the standard form of division.

* It was during the reign and possibly at the command of the caliph Al Mamum (A.D. 813–833) that Mohammed ben Musa, a scholar from Khowarism on the eastern border of the Islamic empire, wrote the book *Hisāb al-Diābr wa'l-Mukabāla* that introduced western Europe to the method of calculation subsequently known (from al-Diābr in the title) as algebra. An Arabic manuscript copy (A.D. 1342) in the Bodleian, translated by F. Rosen (1831), contains a preface in which the author refers to his treatise as " a short work on calculating by completion and reduction, confined to what is easiest and most useful in arithmetic such as men constantly require in cases of inheritance, legacies, partition, law suits and trade . . . the measuring of land, the digging of canals, geometrical computations . . . ".

† The Sanskrit texts of the twelfth and eighteenth chapters were found and translated by Colebrooke : they are included in the volume containing his translations of Bhascara's Viga-ganita and Lilavati (*c.* A.D. 1120).

A graduated course of instruction in this subject implies, of course, a series of problems that are increasingly difficult to reduce to quadratic equation form; naturally, therefore, we expect to see the Babylonian pupil confronted with the need to make preliminary calculations in order to produce the coefficients required for a solution.

B.M. 13901, *No.* 14.

Text: "The sum of my two squares is 1525. The side of one is two-thirds the length of the other, plus 5."

$$(x^2+y^2)=1525 \text{ and } y=(2/3)x+5$$
$$\therefore\ (13/9)x^2+(20/3)x=1500$$

Text	*Formula*
Write down 1, (2/3), 5, (2/3)	
$5^2=25$	
$1525-25=1500$	c
$1^2=1$	
$(2/3)^2=(4/9)$	
$1+(4/9)=(13/9)$	a
$(13/9)1500=(6500/3)$	ac
$(2/3)5=(10/3)$	$(b/2)$
$(10/3)^2=(100/9)$	$(b/2)^2$
$(100/9)+(6500/3)=(19600/9)$	$(b/2)^2+ac=Z$
$=(140/3)^2$	$=(\surd Z)^2$
$(140/3)-(10/3)=(130/3)$	$\surd Z-(b/2)=ax$
$(9/13)(130/3)=30$	$(1/a)ax=x$
$(2/3)30=20$	$(2/3)x$
$(20+5)=25$	$(2/3)x+5=y$

Having obtained ax the text says the reciprocal of a, that is of (13/9), is unknown * and asks "what must I put to (13/9) to make (130/3)?" Answer 30.

In the following problem (from a tablet in the Museum of Strasbourg University) the rectangle had an area of 375 sar (= 375 sq. gar ; gar = pole) and a width of $30x$ where x is the unknown original length (in gar) of the measuring rod. The rod is then shortened by 1 kus (kus = cubit = (1/12) gar) and the length of the rectangle is 60 rods in terms of this shortened rod. The length, therefore, is $60(x-(1/12))=60x-5$ gar and the equation of area is:

$$1800x^2-150x=375 \text{ sar}$$
$$\therefore\ 1800x^2=150x+375$$

Text	*Formula*
Write down (1/12) and 30.	
Write down 1, the original length of the rod.	
$60\times 1=60$ = "False side"	
$30\times 1=30$ = "False end"	
$30\times 60=1800$ = "False area"	a
$1800\times 375=675000$	ac

* The reciprocal of (13/9) is recorded as unknown because (1/13) in sexagesimal notation is the *recurring* fraction 0 ; 4, 36, 55, 23 and was omitted from the standard tables of reciprocals with which the pupils would be familiar. All recurring fractions were omitted from such tables but this does not necessarily mean that mathematicians were unaware of abbreviated values for them.

$(1/12)60 = 5$		
$5 \times 30 = 150$		b
$(1/2)150 = 75$		$(b/2)$
$75^2 = 5625$		$(b/2)^2$
$5625 + 675000 = 680625$		$(b/2)^2 + ac = Z$
$= 825^2$		$= (\sqrt{Z})^2$
$825 + 75 = 900$		$\sqrt{Z} + (b/2) = ax$
Reciprocal of $1800 = (1/1800)$		$(1/a)$
$(1/1800)900 = (1/2) =$ rod		$(1/a)ax = x$

Thus, the original and the shortened lengths of the measuring rod were 6 and 5 kus respectively, making the width × length of the rectangle :

$$(30 \times 6)(60 \times 5)/(12 \times 12) = 15 \times 25 = 375 \text{ sq. gar} = 375 \text{ sar.}$$

It is from texts such as these that some of the Babylonian metrological ratios become apparent.* Mathematically the above solution is particularly interesting because the expressions false side and false end for the products 60×1 and 30×1 respectively seem to imply that unity was here used symbolically ; presumably, the pupil received an oral explanation.

The next example is one of the texts that attempt to arouse interest by realism, its problem is expressed in terms of a military ramp for attacking an enemy city. Such a structure, of course, must attain an appropriate height in relation to the wall to be scaled but in problem No. 25 on B.M. 85194 the ramp is unfinished ; it has been built only to height $h = 36$ kus where the gap between ramp and wall is $g = 8$ gar. The final height x kus and total length of base y gar are unknown, but the data include $B =$ constant breadth $= 6$ gar and $V =$ volume of soil required for the whole ramp $= 5400$ sar. This sar of volume was not the cubic gar, it was the mixed unit gar^2 kus (reflecting the practice of expressing horizontal and vertical measurements in gar and kus respectively) and the calculations proceed on this basis.

The geometry of the ramp (possibly the pupils had to draw its side elevation on their tablets?) incorporates similar right-angle triangles which show :

$$(x/y) = (x-h)/g \; ; \text{ and } xy = 2(V/B)$$

$\therefore$ by multiplication : $x^2 = 2(V/B)(x/g) - 2(V/B)(h/g)$

$$\therefore x^2 + 2(V/Bg)h = 2(V/Bg)x$$

$$\therefore a = 1 \; ; \; b = 2(V/Bg) = 225 \; ; \; c = 2(V/Bg)h = 8100$$

$$\therefore x^2 + 8100 = 225x.$$

B.M. 85194, *No.* 25.

Text	*Formula*
Reciprocal of $6 = (1/6)$	$(1/B)$
$(1/6)5400 = 900$	(V/B)
Reciprocal of $8 = (1/8)$	$(1/g)$
$(1/8)900 = 112{\cdot}5$	$(V/Bg) = (b/2)$
$2 \times 112{\cdot}5 = 225$	b
$225 \times 36 = 8100$	$bh = c$
$112{\cdot}5^2 = 12656{\cdot}25$	$(b/2)^2$
$(12656{\cdot}25 - 8100) = 4556{\cdot}25$	$(b/2)^2 - c = Z$
$= 67{\cdot}5^2$	$(\sqrt{Z})^2$
$(112{\cdot}5 - 67{\cdot}5) = 45$	$(b/2) - \sqrt{Z} = x$
$(1/2)45 = 22{\cdot}5$	$(x/2)$
Reciprocal of $22{\cdot}5 = (1/22{\cdot}5)$	$(2/x)$
$(1/22{\cdot}5)900 = 40$	$(2/x)(V/B) = y$

* *Historical Metrology* by A. E. Berriman (Dent. 1953).

Check :

$5400 = \text{Volume}$	V
$22{\cdot}5 \times 40 = 900$	$(x/2)y = (V/B)$
$900 \times 6 = 5400$	$(V/B)B = V$

$\therefore$ Height $= x = 45$ kus ; Length of base $= y = 40$ gar
Volume $= (1/2)Bxy = 5400$ gar^2 kus $= 5400$ sar

Reverting to the B.M. tablet 13901 with its progressive course of instruction, here is the penultimate problem which evidently was intended to remind the pupil that the standard procedure had shown him how the addition of a unit would change $(x^2 + 2x)$ into $(x + 1)^2$ and to tell him to apply this knowledge by borrowing a unit in order to solve $x^2 + 4x = (25/36)$.

B.M. 13901, *No.* 23.

"The area of my square plus the sum of its sides is (25/36)."

$$x^2 + 4x = (25/36)$$

Text	*Formula*
Write down 4, the number of sides	Coefficient of x
Reciprocal of $4 = (1/4)$	
$(1/4)(25/36) = (25/144)$	$(x/2)^2 + x$
Add one unit $= (169/144)$	$(x/2)^2 + x + 1$
$= (13/12)^2$	$((x/2) + 1)^2$
It is the square of $(1/12) + 1$	
Subtract the unit $= (1/12)$	$(x/2)$
Multiply by $2 = (1/6)$	x

In the ninth problem on B.M. 13901 the pupil is introduced to a simultaneous quadratic and instructed to make use of a relationship that is reflected in Euclid II. 9 ; here, therefore, I use explanatory symbols that would be applicable to an algebraic representation of Euclid's relevant geometric propositions. Thus :

Let a straight line be divided into two equal parts p, and also into two unequal parts x and y ; let q be the line between the points of section. Then :

$$x = (p + q) \text{ and } y = (p - q) \qquad \text{by definition}$$
$$\therefore (x + y) = 2p \qquad \therefore p = (1/2)(x + y)$$
$$\therefore (x - y) = 2q \qquad \therefore q = (1/2)(x - y)$$

$$(x + y)^2 = x^2 + y^2 + 2xy = (2p)^2 \qquad \text{Euc. II. 4.}$$
$$x^2 + y^2 = (2p)^2 - 2xy$$

$$xy + q^2 = p^2 \qquad \text{Euc. II. 5.}$$
$$\therefore xy = p^2 - q^2$$

$$x^2 + y^2 = 2(p^2 + q^2) \qquad \text{Euc. II. 9.}$$

Let : $P = (1/2)(x^2 + y^2) = (p^2 + q^2)$
And : $Q = (1/2)(x^2 - y^2)$

Then : $x^2 = (P + Q) \quad y^2 = (P + Q) \quad Q^2 = P^2 - (xy)^2$

B.M. 13901, *No.* 9.

Text : "The sum of my two squares is 1300. The side of one exceeds the side of the other by 10."

$$(x^2 + y^2) = 1300 \text{ and } (x - y) = 10$$

Text	*Formula*
$(1/2)1300 = 650$	$(1/2)(x^2 + y^2) = P$
$(1/2)10 = 5$	$(1/2)\ (x - y) = q$
$5^2 = 25$	q^2
$620 - 25 = 625 = 25^2$	$(P - q^2) = p^2$
$25 + 5 = 30$	$(p + q) = x$
$25 - 5 = 20$	$(p - q) = y$

B.M. 13901, *No.* 12.

Text : " The sum of my two squares is 1300. The product of their sides is 600."

$$(x^2 + y^2) = 1300 \text{ and } xy = 600$$

Text	*Formula*
$(1/2)1300 = 650$	$(1/2)(x^2 + y^2) = P$
$650^2 = 422500$	P^2
$600^2 = 360000$	xy^2
$650^2 - 600^2 = 62500 = 250^2$	$P^2 - (xy)^2 = Q^2$
$650 + 250 = 900 = 30^2$	$P + Q = x^2$
$30 = \text{side}$	x
$650 - 250 = 400 = 20^2$	$P - Q = y^2$
$20 = \text{side}$	y

In the following problem relating to a pit the length and breadth are x and y gar respectively and the area $xy = 1$ sq. gar $= 1$ area sar. For a reciprocal relationship of x and y such as this the texts use the words igu and igibu for the length and breadth respectively. The depth $z = (x + y)$ gar $= 12(x + y)$ kus and the volume of soil excavated is $xyz = 26$ gar^2 kus $= 26$ volume sar. Obviously, therefore, $z = 26$ kus. And $(26/12) = (x + y)$ gar.

$$xy = 1 \text{ sq. gar} ; \; xyz = 26 \text{ gar}^2 \text{ kus} \quad \therefore z = 26 \text{ kus} ; \; (x + y) = (26/12) \text{ gar.}$$

B.M. 85200, *No.* 16.

Text	*Formula*
Reciprocal of $12 = (1/12)$	Gar per kus
$(1/12)26 = (26/12)$	$(x + y)$
$(1/2)(26/12) = (13/12)$	$(1/2)(x + y) = p$
$(13/12)^2 = (169/144)$	p^2
$(169/144) - 1 = (25/144) = (5/12)^2$	$(p^2 - xy) = q^2$
*$(13/12) \pm (5/12) = (3/2)$ and $(2/3)$	$(p \pm q) = x$ and y
$(3/2) = $ igu	x
$(2/3) = $ igibu	y
depth $= 26$	z

In the next problem (from a tablet in the Berlin Museum) the volume xyz is that of a brick wall containing 9 sar of bricks and the opening line of the calculation in the text shows that the sar of bricks occupied (4/9) sar of volume.† The volume of the wall, therefore, was $xyz = 4$ sar $= 4$ gar^2 kus and as its height z is given as 12 kus the product

$$\text{length} \times \text{thickness} = xy = (4/12) = (1/3) \text{ sq. gar.}$$

The text states that the length + breadth $= (x + y) = (13/6)$ gar.

* The text gives the instruction to add and subtract.

† In the Babylonian linear scale, 1 gar $= 12$ kus $= 60$ gin $= 360$ shusi. Volume sar $=$ gar^{2}kus $= 360^2 \times 30$ cu. shusi. $\therefore$ Sar of bricks $= 720 \times 2400$ cu. shusi. On other evidence Neugebauer interprets the sar of bricks numerically as 60 dozen ; a representative brick, therefore, could be 2400 cu. shusi in volume or say, a foot (20 shusi) square by a gin (6 shusi) thick, but there were bricks of many different sizes. The

VAT. 6596, *No*. 3.

Text	*Formula*	
Reciprocal of $(9/4)=(4/9)$	Vol. sar per sar of bricks	
$(4/9)9=4$	xyz	gar^2 kus
Reciprocal of $12=(1/12)$	$(1/z)$	z in kus
$(1/12)4=(1/3)$	$(1/z)xyz \quad xy=$	sq. gar
$(1/2)(13/6)=(13/12)$	$(1/2)(x+y)=p$	
$(13/12)^2=(169/144)$	p^2	
$(169/144)-(1/3)=(121/144)$ $=(11/12)^2$	$(p^2-xy)\quad =q^2$	
$(13/12)+(11/12)=2$	$(p+q)=x=2$	gar
$(13/12)-(11/12)=(1/6)$	$(p-q)=y=(1/6)$	gar

Several tablets introduce wage rates and time into their xyz volume problems. Here, for example, is a synopsis of the data and calculations on a tablet in the Yale Babylonian Collection; the job of work (excavation?) is called ki-la.

YBC. 4663, *No*. 7.

Daily	*Total*
$w=$ wages $=6$ se of silver	$W=9$ gin $=9\times 180$ se
	$\therefore\ T=(W/w)=270$ days
$v=$ work $=10$ gin $=(1/6)$ sar	$\therefore\ V=vT=xyz$ gar^2 kus
	$=(1/6)270=45$ gar^2 kus

Also given:

Depth $=z=(1/6)$ gar $=6$ kus $\quad\therefore\ xy=(45/6)\ \text{gar}^2$
(Length + breadth) $=(x+y)=(13/5)$ gar

Text	*Formula*	
Reciprocal of wages $=(1/6)$		
$(1/6)(9\times 180)=270$	$(W/w)=T$	days
$(1/6)270=45$	$vT=xyz$	gar^2 kus
Reciprocal of depth $=(1/6)$	$(1/z)$	
$(1/6)45=(15/2)$	$(xyz/z)=xy$	sq. gar
$(1/2)(13/2)=(13/4)$	$(1/2)(x+y)=p$	
$(13/4)^2=(169/16)$	p^2	
$(169/16)-(15/2)=(49/16)$	$(p^2-xy)=q^2$	
$\sqrt{(49/16)}=(7/4)$	q	
$(13/4)+(7/4)=5$	$(p+q)=x$	gar
$(13/4)-(7/4)=(3/2)$	$(p-q)=y$	gar

In the following example, based on a tablet in the Berlin Museum, a farmer sowed two fields x and y at 4 and 3 gur (gur $=300$ qa) of seed per bur of area respectively and the amount of seed sown on x exceeded that on y by 500 qa. Total area $(x+y)=1800$ sar $=1$ bur.

Let $R=$ rate of sowing on $x=(1200/1800)=(2/3)$ qa per sar
$r=$ rate of sowing on $y=(\ 900/1800)=(1/2)$ qa per sar
$\therefore\ (Rx-ry)=R(p+q)-r(p-q)=p(R-r)+q(R+r)=500$ qa.

Sumerian shusi represented by the average length of the divisions of the linear scale on Gudea's statue (*c.* 2175 B.C.) in the Louvre is 0·66 inch; surprisingly, therefore, the English pole = 10 Sumerian cubits and the medieval foot of 13·2 inches (which Petrie called the most usual English unit) reflects the Sumerian foot, it measures (1/15) pole. Equally surprising is the fact that the principal unit of the linear scale engraved on a fragment of shell found at Mohenjo-daro in the Indus Valley measures 1·32 inches = 2 Sumerian shusi; I call this the Indus inch.

VAT. 8389, *No*. 1.

Text	*Formula*
1800 = bur	sar per bur
1200 = seed per bur	Qa per bur on x
1800 = second bur	Sar per bur
900 = seed per bur	Qa per bur on y
500 = difference in seed	$(Rx - ry) = p(R - r) + q(R + r)$
1800 = combined area	$(x + y)$
(1/2)1800 = 900	$(1/2)(x + y) = p$
900 and 900	p and p
Reciprocal = (1/1800)	(bur/sar)
(1/1800)1200 = (2/3)	R
(2/3)900 = 600	Rp
Reciprocal = (1/1800)	(bur/sar)
(1/1800)900 = (1/2)	r
(1/2)900 = 450	rp
(600 − 450) = 150	$(Rp - rp) = p(R - r)$
(500 − 150) = 350	$q(R + r)$
(2/3) + (1/2) = (7/6)	$(R + r)$
(6/7)350 = 300	$q(R + r)/(R + r) = q$
900 + 300 = 1200	$(p + q) = x = 1200$ sar = (2/3) bur
900 − 300 = 600	$(p - q) = y = 600$ sar = (1/3) bur

Expressing areas in bur and the 500 qa difference in seed as (5/3) gur, the equation is :

$$4x - 3y = (5/3) \text{ and } (x + y) = 1 \quad \therefore\ p = (1/2)(x + y) = (1/2)$$
$$\therefore\ 4(p + q) - 3(p - q) = p(4 - 3) + q(4 + 3) = p + 7q = (5/3) \quad \therefore\ q = (1/6)$$
$$\therefore\ x = (p + q) = (2/3) \text{ and } y = (p - q) = (1/3)$$

A. E. B.

1864. This was the period (1500 to 1505) when Dürer's workshop (for he was now a master-painter and took in apprentices) produced the *Lamentation over the Dead Body of Christ*, which the Holzschuher family hung from a pillar in the Church of St. Sebald. Also of this date are his two most important works, the *Baumgärtner Altar*, for the Church of St. Catherine, and an *Adoration of the Magi*, commissioned by Frederick the Wise. Dürer had never before set himself so seriously to tackle the problems of the new science of perspective. The two latter paintings, in which he piled building upon building and ruin upon ruin, showed his complete mastery of the formal structure of a large composition. In the complication of architectural planes and arches his students must have found wonderful material with which to study the new branch of art.

The same lively interest must have attached to the drawings for the *Green Passion*, executed in 1504, in which Dürer went out of his way to pose his figures in the most difficult attitudes, and the series of twenty woodcuts on the *Life of the Virgin*, which he began in 1502. Here again he filled his picture with architectural forms and solved problems of perspective with the most brilliant display of virtuosity. One drawing in particular, the *Presentation in the Temple*, was reproduced in a treatise on perspective by Jean Pelerin, *De Artificiali Perspectiva* (1504 and 1509). Incidentally, 1509 was the year when Luca Pacioli published his *De Divina Proportione*, which was illustrated with sixty drawings by Leonardo da Vinci.—Pierre Descargues, *Dürer*. [Per Mr. E. H. Lockwood.]

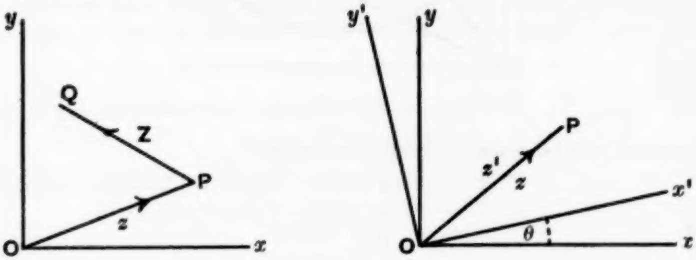
COMPLEX NUMBER AND TWO-DIMENSIONAL MECHANICS. I

BY A. BUCKLEY

THE approach to plane mechanics by vector methods is often considered to be an unnecessarily sophisticated one and it is true that the preliminary work in vector algebra is lengthy and out of keeping with the elegance achieved. The natural tool, it has been said, is the complex number, and I shall obtain some of the fundamental results by using only complex number theory and vector addition.

It will be necessary to obtain complex forms for the moment of a vector about a point, and for the "work done" by a vector in a small displacement. Consider the vector **PQ** represented by Z associated with the point represented by z (Fig. 1). By expanding the product $Z\bar{z}$, where $\bar{z}$ is the conjugate complex of z, it follows that

(*a*) $\mathscr{I}\,(Z\bar{z})$ is the moment of Z about O ;
(*b*) $\mathscr{R}\,(Z\,.\,d\bar{z})$ is the work done by Z in displacing P through dz.



FIGS. 1 AND 2

We shall require one other result of importance, relating the rate of change of a vector to moving axes in the plane. If the point P referred to the axes $Ox'y'$ is z' (Fig. 2), then

$$z' = ze^{-i\theta}, \quad z = z'e^{i\theta},$$

and on differentiation with respect to time we have

$$\dot{z} = \dot{z}'\,e^{i\theta} + iz'\dot{\theta}\,e^{i\theta}.$$

By putting $\theta = 0$ we obtain on the right-hand side the rate of change of the vector z referred to axes $Ox'y'$ rotating with angular velocity $\dot{\theta}$ and coinciding instantaneously with the fixed axes Oxy. Then

(*c*) $$\dot{z} = \dot{z}' + iz'\dot{\theta},$$

and this is the moving axes result for two-dimensional motion.

Kinematics

1. The result (*c*) enables us to write down the components of velocity and acceleration in Cartesian and polar coordinates ; we have, applying (*c*) to $\dot{z}$,

$$\ddot{z} = \ddot{z}' + iz'\ddot{\theta} + 2i\dot{z}'\dot{\theta} - z'\dot{\theta}^2,$$

and these results will give the corresponding formulae in Cartesians if we substitute $z' = x' + iy'$, and in polars if we put $z' = r$.

2. As a further illustration of the use of (*c*) we shall obtain the body-centrode and space-centrode property.

N

Let a lamina be moving in its own plane and let a denote the position of a given point A of the lamina and z the position of any other point P (Fig. 3), so that

$$\mathbf{AP} = z - a$$

and
$$\dot{z} = \dot{a} + i(z-a)\omega, \qquad \text{.......(1)}$$

where ω is the angular velocity of the lamina. The instantaneous centre I

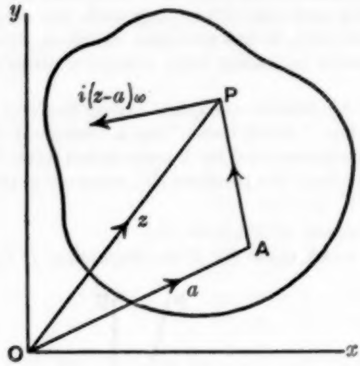


FIG. 3

of the motion will be given by $\dot{z} = 0$, that is, by

$$z = a + i\dot{a}/\omega \qquad \text{.......(2)}$$

so that
$$\dot{z} = \frac{d}{dt}(a + i\dot{a}/\omega)$$

and this is the velocity with which I describes the space-centrode referred to the fixed axes Oxy. Now the position of I referred to axes fixed in the body, that is, relative to A, is, using (1)

$$z - a = i\dot{a}/\omega,$$

and to express the *velocity of this point in the body* but referred to *axes fixed in space* we use the result (c). In this case the axes fixed in space "sweep through the body" with angular velocity $-\omega$, so that the velocity with which I describes the body-centrode is

$$\frac{d}{dt}(i\dot{a}/\omega) + i(i\dot{a}/\omega)(-\omega)$$
$$= \frac{d}{dt}(a + i\dot{a}/\omega) = \dot{z},$$

and the well-known property follows at once.

Dynamics

1. In dealing with the general motion of a plane system of particles, let the suffix r refer to a typical particle, so that

$$m_r\ddot{z}_r = Z_r + Z_r', \quad (r = 1, 2, \ldots, n)$$

where Z_r and Z_r' are the resultants of the external and internal forces respectively acting on the particle. Summing for the whole system, so that the summation sign refers to a sum over r from 1 to n, we have, since $\Sigma Z_r' = 0$,

$$\Sigma m_r\ddot{z}_r = \Sigma Z_r.$$

If the typical particle be at ρ_r relative to the centroid γ of the whole system, we can write

$$\Sigma m_r(\ddot{\gamma}+\ddot{\rho}r)=\Sigma Z_r$$

and hence

$$M\ddot{\gamma}=\Sigma Z_r,$$

where M is the total mass of the system, since $\Sigma m_r\rho_r=0$.

By taking moments for all the particles about the origin, we have, in virtue of (a),

$$\mathscr{I}\,\Sigma m_r\ddot{z}_r\bar{z}_r=\mathscr{I}\,\Sigma Z_r\bar{z}_r,\quad(\text{since } \mathscr{I}\,\Sigma Z_r'\bar{z}_r=0)$$

that is $$\mathscr{I}\,\Sigma m_r(\ddot{\gamma}+\ddot{\rho}_r)(\bar{\gamma}+\bar{\rho}_r)=\mathscr{I}\,\Sigma Z_r(\bar{\gamma}+\bar{\rho}_r)$$

or $$\mathscr{I}\,M\ddot{\gamma}\bar{\gamma}+\mathscr{I}\,\Sigma m_r\ddot{\rho}_r\bar{\rho}_r=\mathscr{I}\,\Sigma Z_r\bar{\gamma}+\mathscr{I}\,\Sigma Z_r\bar{\rho}_r$$

and the first terms on each side cancel in virtue of the previous result, giving the usual result

$$\mathscr{I}\,\Sigma m_r\ddot{\rho}_r\bar{\rho}_r=\mathscr{I}\,\Sigma Z_r\bar{\rho}_r. \qquad \text{.................................(1)}$$

If $\dot{\theta}$ be the angular velocity of the system when it constitutes a plane lamina, we can use fixed axes in the body at the centroid and obtain the velocity of a point in space referred to these axes by using (c). In the previous notation

$$\dot{\rho}_r=i\rho_r'\dot{\theta}$$

where ρ' is now constant, and using (c) again,

$$\ddot{\rho}_r=i\rho_r'\ddot{\theta}+i(i\rho_r'\dot{\theta})\dot{\theta}.$$

Substituting in (1) and remembering that $\rho_r=\rho_r'$, we have

$$\mathscr{I}\,(\Sigma m_r i\rho_r'\bar{\rho}_r'\ddot{\theta}-\Sigma m_r\rho_r'\bar{\rho}_r'\dot{\theta}^2)=\mathscr{I}\,\Sigma Z_r\bar{\rho}_r'$$

that is,

$$\Sigma m_r\mid\rho_r'\mid^2\ddot{\theta}=\mathscr{I}\,\Sigma Z_r\bar{\rho}_r',\ (\text{since } \Sigma m_r\rho_r'\bar{\rho}_r'\dot{\theta}^2 \text{ is real})$$

or, in the usual notation,

$$I_G\ddot{\theta}=N.$$

2. The independence of the kinetic energies of translation and rotation is proved in a similar manner. We have

$$\begin{aligned}2T&=\Sigma m_r\mid\dot{z}_r\mid^2=\Sigma m_r\dot{z}_r\dot{\bar{z}}_r\\&=\Sigma m_r(\dot{\gamma}+\dot{\rho}_r)(\dot{\bar{\gamma}}+\dot{\bar{\rho}}_r)\\&=M\dot{\gamma}\dot{\bar{\gamma}}+\Sigma m_r\dot{\rho}_r\dot{\bar{\rho}}_r\end{aligned}$$

the other terms vanishing since $\Sigma m_r\rho_r$ etc. are zero. The result can clearly be written

$$2T=M\mid\dot{\gamma}\mid^2+\Sigma m_r\mid\dot{\rho}_r\mid^2,$$

and if the particles form a lamina we can use moving axes as in (1), when the equation takes the usual form

$$2T=M\mid\dot{\gamma}\mid^2+I_G\dot{\theta}^2.$$

3. Multiplying the equations of motion by $\dot{\bar{z}}_r$ and summing for all particles, we have

$$\Sigma m_r\ddot{z}_r\dot{\bar{z}}_r=\Sigma Z_r\dot{\bar{z}}_r+\Sigma Z_r'\dot{\bar{z}}_r,$$

that is,

$$\mathscr{R}\,\Sigma m_r\ddot{z}_r\dot{\bar{z}}_r=\mathscr{R}\,\Sigma Z_r\dot{\bar{z}}_r+\mathscr{R}\,\Sigma Z_r'\dot{\bar{z}}_r,$$

or

$$\tfrac{1}{2}(d/dt)\ \Sigma m_r \dot{z}_r \dot{\bar{z}}_r = \mathcal{R} Z_r \dot{\bar{z}}_r + \mathcal{R}\ \Sigma Z_r' \dot{\bar{z}}_r,$$

so that, since the second term on the right vanishes,

$$\tfrac{1}{2}\frac{d}{dt}(\Sigma m_r \mid \dot{z}_r \mid^2) = \mathcal{R}\ \Sigma Z_r \dot{\bar{z}}_r,$$

and thus on integration

$$2T = \int \mathcal{R} Z_r\ d\bar{z}_r\ ;$$

the right hand term is the work done, from (*b*) and thus the principle of the conservation of energy follows.

4. The equations of impulsive motion can be obtained as follows. Using the obvious notation, we have, for a single particle

$$m_r(\dot{z}_{r1} - \dot{z}_{r0}) = Z_r + Z_r'.$$

Summing for the whole system, since $\Sigma Z_r' = 0$, we have

$$\Sigma m_r(\dot{z}_{r1} - \dot{z}_{r0}) = \Sigma Z_r.$$

Writing $z_{r0} = \gamma_0 + \rho_{r0}$ where of course $z_{r1} = z_{r0}$, we have

$$\Sigma m_r(\dot{\gamma}_1 + \dot{\rho}_{r1} - \dot{\gamma}_0 - \dot{\rho}_{r0}) = \Sigma Z_r,$$

so that

$$M(\dot{\gamma}_1 - \dot{\gamma}_0) = \Sigma Z_r.$$

By taking moments for all particles about the origin, using (*a*), we then have

$$\mathcal{I}\ \Sigma m_r(\dot{\gamma}_1 + \dot{\rho}_{r1} - \dot{\gamma}_0 - \dot{\rho}_{r0})(\bar{\gamma}_0 + \bar{\rho}_{r0}) = \mathcal{I}\ \Sigma Z_r \bar{z}_{r0},$$

and simplifying

$$\mathcal{I} M(\dot{\gamma}_1\bar{\gamma}_0 - \dot{\gamma}_1\bar{\gamma}_0) + \mathcal{I}\ \Sigma m_r(\dot{\rho}_{r1}\bar{\rho}_{r0} - \dot{\rho}_{r0}\bar{\rho}_{r0}) = \mathcal{I}\ \Sigma Z_r(\bar{\gamma}_0 + \bar{\rho}_{r0}).$$

The first terms on each side will cancel in view of the above and we have, since $\rho_{r1} = \rho_{r0}$,

$$\mathcal{I}\ \Sigma m_r(\dot{\rho}_{r1}\bar{\rho}_{r1} - \dot{\rho}_{r0}\bar{\rho}_{r0}) = \mathcal{I}\ \Sigma Z_r \bar{\rho}_{r0},$$

and using the moving axes result (*c*) we have, for a plane lamina,

$$\mathcal{I}\ \Sigma m_r(i\rho_{r1}\dot{\theta}_1\bar{\rho}_{r1} - i\rho_{r0}\dot{\theta}_0\bar{\rho}_{r0}) = \mathcal{I}\ \Sigma Z_r \bar{\rho}_{r0},$$

that is,

$$(\dot{\theta}_1 - \dot{\theta}_0)\ \Sigma m_r \mid \rho_r \mid^2 = \mathcal{I}\ \Sigma Z_r \bar{\rho}_{r0},$$

where $\rho_r = \rho_{r0} = \rho_{r1}$, and thus

$$I_G(\dot{\theta}_1 - \dot{\theta}_0) = \Gamma,$$

where Γ is the moment of the external impulses about the centroid.

Analytical statics

1. Consider a small translation δa of a plane system coupled with a small rotation $\delta\theta$. Then if δz_r denotes the displacement of the point P of the system relative to the fixed axes Oxy, we see from Fig. 4 that

$$\delta z_r = z_r' - z_r = \delta a + z_r e^{i\delta\theta} - z_r.$$

If the work done in this small displacement is δW, then

$$\begin{aligned}\delta W &= \mathcal{R}\ \Sigma Z_r\{\delta\bar{a} + (e^{-i\delta\theta} - 1)\bar{z}_r\}\\ &= \mathcal{R}(\Sigma Z_r)\ \delta\bar{a} - \mathcal{R} i\ \delta\theta\ \Sigma Z_r \bar{z}_r,\end{aligned}$$

and if $\delta a = \delta x + i\ \delta y$, we have, in the usual notation,

$$\delta W = (\Sigma X_r)\ \delta x + (\Sigma Y_r)\ \delta y + N\ \delta\theta$$

and the principle of virtual work can be enunciated.

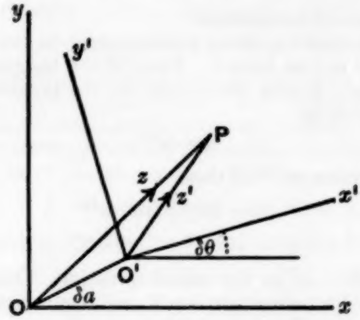


FIG. 4

2. The equation of the line of action of the resultant of a plane system of forces acting at given points is obtained by taking moments about any point ζ on the line of action. We have then

$$\mathscr{I}(\Sigma Z_r)\bar{\zeta} = \mathscr{I}\,\Sigma Z_r \bar{z}_r,$$

that is, putting $\zeta = x + iy$,

$$x(\Sigma Y_r) - y(\Sigma X_r) - N = 0,$$

and this is the equation of the line of action.

It is interesting to note in this connection that, if all the forces be turned through an angle θ about their fixed points of application, then the new resultant is

$$\Sigma e^{i\theta} Z_r = e^{i\theta}\,\Sigma Z_r,$$

which is the original resultant turned through an angle θ and unaltered in magnitude. Further it is easy to show that

$$\mathscr{I}\,\Sigma(iZ_r)\bar{z}_r = \mathscr{R}\,\Sigma Z_r \bar{z}_r,$$

so that, if ζ be the point of intersection of the resultants of the two systems Z_r and iZ_r, we can write

$$\bar{\zeta}\,\Sigma Z_r = \Sigma Z_r \bar{z}_r, \quad \text{.................................}(1)$$

or, in an obvious notation,

$$\bar{\zeta} = (M + iN)/\Sigma Z_r$$

Now if the forces be turned through an angle θ as above and if ζ' be the point of intersection of the corresponding resultants, we have

$$\bar{\zeta}'\,\Sigma e^{i\theta} Z_r = \Sigma e^{i\theta} Z_r \bar{z}_r$$

which clearly reduces to (1) and therefore ζ is the astatic centre of the system

A. B.

II

BY F. CHORLTON

Contemporary writers of textbooks on electricity and magnetism and on hydrodynamics make considerable use of the complex variable in two-dimensional problems. It seems strange that two-dimensional dynamics is not taught in this way, even though the application of vectors to three-dimensional dynamics has now become respectable. Mr. Buckley's article preceding this has shown how the basic results may be established. The following examples are further indications of how this powerful technique may be profitably employed in two-dimensional kinematics. The methods are essentially labour-saving.

1. *Tangential and normal components*

Let P be a particle moving along a fixed curve in the (x, y) plane, and suppose that its velocity is v at time t. Then if the tangent to the curve at this instant makes an angle ψ with the x-axis (in the positive sense), the complex velocity w of P is given by

$$w = ve^{i\psi}. \qquad (1)$$

The complex acceleration of P is thus

$$\begin{aligned}\dot{w} &= \dot{v}e^{i\psi} + (v\dot{\psi})\, ie^{i\psi}\\ &= \dot{v}e^{i\psi} + (\kappa v^2) ie^{i\psi}, \qquad (2)\end{aligned}$$

since $\dot{\psi} = (d\psi/ds)(ds/dt) = \kappa v$ in the usual notation. Thus the tangential and normal components of acceleration are $\dot{v}$, κv^2 respectively.

2. *Motion of a point fixed on periphery of a rolling circle*

In Fig. 1 P is a point fixed on the circumference of a circle and initially in contact at O with the line Ox along which the circle rolls. At any time t, M is in contact with Ox and P is in the position shown, where $\angle PCM = \phi$, C being the centre of the circle. Then if the radius of the circle is a, $OM = \text{arc } PM = a\phi$, and the complex coordinate z of P is clearly

$$\begin{aligned}z &= a\phi + ai - a \exp\left(\tfrac{1}{2}\pi i - \phi i\right)\\ &= a\phi + ai - ai \exp(-i\phi) \qquad (3)\end{aligned}$$

From (3) the cartesian coordinates (x, y) are easily found as

$$x = a(\phi - \sin\phi), \quad y = a(1 - \cos\phi).$$

Differentiating (3), the complex velocity is found to be

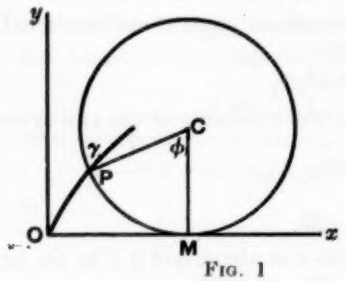


FIG. 1

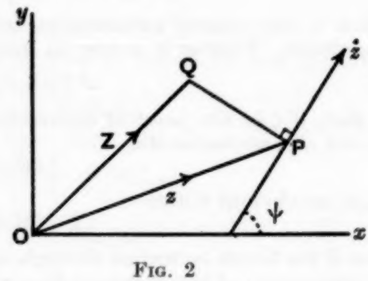


FIG. 2

$$PQ = Ri \exp(i\psi)$$

$$\dot{z} = a\dot{\phi} - a\dot{\phi} \exp(-i\phi) = (2a\dot{\phi} \sin \tfrac{1}{2}\phi) i \exp(-\tfrac{1}{2}i\phi) \qquad (4)$$

This is a vector of magnitude $2a\dot{\phi} \sin \frac{1}{2}\phi$ and direction $\frac{1}{2}\pi - \frac{1}{2}\phi$ to the axis of x (in the positive sense). The instantaneous centre is found from $\dot{z} = 0$, that is,

$$2a\dot{\phi} \sin \tfrac{1}{2}\phi = 0.$$

Since $\dot{\phi} \neq 0$, $\phi = 0, \ \pm 2\pi, \ \pm 4\pi, \ldots$, that is, P is at rest only when it is in contact with Ox. Thus in Fig. 1, M is the instantaneous centre.

The radius of curvature of the locus of P is easily deduced as follows. The direction ψ of the locus of P is

$$\psi = \tfrac{1}{2}\pi - \tfrac{1}{2}\phi. \qquad (5)$$

Let Q be the point at a distance R along the normal to the locus of P (Fig. 2). Then the complex coordinate Z of Q is

$$Z = z + Ri \exp i\psi = z - R \exp(-\tfrac{1}{2}i\phi) \qquad (6)$$

Hence $$Z = \dot{z} - \dot{R} \exp(-\tfrac{1}{2}i\phi) + \tfrac{1}{2}R\dot{\phi}i \exp(-\tfrac{1}{2}i\phi)$$
$$= (2a\dot{\phi} \sin \tfrac{1}{2}\phi + \tfrac{1}{2}R\dot{\phi})i \exp(-\tfrac{1}{2}i\phi) - \dot{R} \exp(-\tfrac{1}{2}i\phi). \qquad (7)$$

If Q is the centre of curvature of the locus of P, then by the involute property, the motion of Q is at right angles to that of P. Now the motion of Z is expressed in (7) by a vector whose components are in the perpendicular directions $\frac{1}{2}\pi - \frac{1}{2}\phi$ and $-\frac{1}{2}\phi$ respectively. Since $\dot{z}$ is in the direction $\frac{1}{2}\pi - \frac{1}{2}\phi$, the coefficient of $i \exp(-\frac{1}{2}i\phi)$ in (7) must vanish. This requires, since $\dot{\phi} \neq 0$,

$$R = -4a \sin \tfrac{1}{2}\phi = -4a \cos \psi \qquad (8)$$

Substituting from (8) into (6), the locus of Q is given by

$$Z = X + iY = z + 4a \sin \tfrac{1}{2}\phi \exp(-\tfrac{1}{2}i\phi)$$
$$= a\phi + ai - ai \exp(-i\phi) + 4a \sin \tfrac{1}{2}\phi \exp(-\tfrac{1}{2}i\phi).$$

Equating real and imaginary parts, we find

$$X = a(\phi + \sin \phi), \quad Y = -a(1 - \cos \phi).$$

This is clearly another cycloid, congruent to the first, but displaced downwards a distance $2a$ and to the right a distance πa. F. C.

AN ENQUIRY INTO THE TEACHING OF MATHEMATICS

The International Commission on the Teaching of Mathematics, which is one of the Commissions set up by the International Mathematical Union, has been conducting an enquiry into the teaching of mathematics between the ages of 16 and 21. A full report has already been published on this aspect of teaching so far as Germany is concerned and a questionnaire was sent to all countries participating in the work of the commission. The British National Committee for Mathematics appointed Dr. E. A. Maxwell to represent Britain on the International Commission and a sub-committee consisting of Dr. Maxwell, Miss M. L. Cartwright, A. P. Rollett and G. L. Parsons to prepare the answers to the questionnaire. This sub-committee also received a good deal of help from the Department of Education of Cambridge University in connection with questions relating to the general system of education. The sub-committee confined their replies for the most part to work done in schools. With the replies were sent various relevant pamphlets published by the Ministry of Education, specimen timetables from various types of schools, a large number of examination papers and syllabuses and reports of the Mathematical Association. The questionnaire was framed for dealing with state systems of education similar to that in Germany and covered the whole educational system. We give only those parts of it relating to the technical aspects of the enquiry.

A summary of the replies was given in a 15-minute report to Section VII of the International Congress of Mathematicians at Amsterdam on September 4th 1954 by Miss M. L. Cartwright.

Fragebogen

3. Besonderheiten und Schwierigkeiten im Unterrichtsstoff und in der Methode.

(a) *Arithmetic:* negative Zahlen, irrationale Zahlen, Verwebung von Potenz und Wurzel, arithmetische und geometrische Reihe, Statistik.

(b) *Algebra:* Funktionen ersten und zweiten Grades, einfache gebrochene Funktionen. Parabel, Hyperbel, Exponentialfunktion, logarithmische Funktion.

(c) *Geometrie :* Methode des Euklid, intuitive Geometrie, Bewegungsgeometrie, Symmetrie, Abbildung von Figuren ; Vermessungen, Stereometrie, darstellende Geometrie, Projektive Geometrie der Kegelschnitte.

(d) *Trigonometry :* Darstellung der Wellen, praktische Anwendungen.

(e) *Analytische Geometrie :* Gerade, Kreis, Kegelschnitte ; Verbindung von projektiver und analytischer Geometrie, Dandelin'sche Schnitte des Kegels.

(f) *Differential- und Integralrechnung :* Grenzwert, Differential-quotient, Differential. Welche Funktionen werden differenziert, welche Kurven diskutiert?

(g) *Sphärische Trigonometrie :* Welche Satze sind abgeleitet : Sinussatz, Kosinussatz, Nepersche Regel? Anwendungen auf Berechnungen der Erde und des Himmels ; Ekliptiksystem.

Extract from the answers to the questionnaire

It should be remarked that various methods are used in practice, and that no detailed instructions on methods are given in this country. The syllabuses of the Examining Bodies give a good indication of the actual scope. All our methods do, however, agree in placing great emphasis on the working of examples by pupils ; this is true both of School and of University.

The Mathematical Association has issued a number of reports on the teaching of particular topics, and these are attached in answer to the questions asked under this heading. The subject matter which is included under *Arithmetic* in this questionnaire would rather be counted in this country as Algebra, merging into Analysis ; Statistics ; and Theory of Numbers. The heading (b) *Algebra* would similarly be called Analytical Geometry or Coordinate Geometry.

The following points of detail are worthy of specific note :

(c) *Geometry.* The study of deductive Geometry is regarded as important, although the Euclidean sequence is not observed. At the present time, the tendency is for detailed proofs of the standard theorems to be taught in class, but, with a few exceptions, to be excluded from examinations. Solid Geometry is studied informally only, and Projective Geometry does not appear in the Schools until the latest stages.

(d) *Trigonometry.* The work in Secondary Grammar Schools is concerned chiefly with the elementary definitions and applications such as appear in problems concerning the triangle and so on. The representation of waves, to which the questionnaire refers, would come chiefly in the Technical Schools or Colleges.

(e) *Analytical Geometry.* The work indicated by the questionnaire is studied in the Mathematical Divisions throughout the Upper School, save that the treatment by actual sections of the cone is not usually adopted.

(g) *Spherical Trigonometry* is not normally a school subject, except where there are special interests, such as an observatory, or where navigation is part of the curriculum.

(h) *Philosophy and History of Mathematics.* Little is done about the Philosophy of Mathematics. The History of Mathematics is (with occasional exceptions) not included as an examination subject, but individual teachers often include a fair amount of History on an informal basis.

(i) *Magazines.* There is no well-established magazine primarily intended for School children, but the *Mathematical Gazette,* to which we refer later, contains matter of interest to teachers and their abler pupils.

(j) *Mechanics.* It should be added that, in this country, Mechanics is regarded as an integral part of Mathematics and not only as part of Physics.

MATHEMATICAL NOTES

2610. *Quadratic polynomials and prime numbers.*

In C. Smith's *Treatise on Algebra*, page 481, following the proof of the theorem that " No rational integral algebraic function can represent prime numbers exclusively " is the observation that

$$\begin{array}{ll} x^2+x+41 & \text{is prime if } x<40 \\ x^2+x+17 & \text{is prime if } x<16 \\ 2x^2+29 & \text{is prime if } x<29 \end{array}$$

the first statement being attributed to Euler, the other two to Barlow. In a Note in *Mathematical Gazette*, Vol. XVII, No. 222 (February 1933), Mr. F. C. Boon says that Fazzari's *Arithmetic* ascribes the first two to Euler and the third to Legendre. However, Dickson's *History of the Theory of Numbers*, Vol. I, Chapter XVIII, credits the second to Euler and the other two to Legendre. The theorem quoted above seems to have been first proved by Euler.

In their *Theory of Numbers*, (1938 edition), page 18, Hardy and Wright quote as " mathematical curiosities " the fact that

$$n^2-n+41 \text{ is prime for } 0 \leqslant x \leqslant 41$$

and

$$n^2-79n+1601 \text{ is prime for } 0 \leqslant x \leqslant 79.$$

It is interesting to pursue some of these curiosities further.

1. Considering the expression x^2+x+A it is found that

$$\begin{array}{ll} x^2+x+1 & \text{is prime for } -4 \leqslant x \leqslant 3 \\ x^2+x+3 & \text{is prime for } -2 \leqslant x \leqslant 1 \\ x^2+x+5 & \text{is prime for } -4 \leqslant x \leqslant 3 \\ x^2+x+11 & \text{is prime for } -10 \leqslant x \leqslant 9 \\ x^2+x+17 & \text{is prime for } -16 \leqslant x \leqslant 15 \\ x^2+x+41 & \text{is prime for } -40 \leqslant x \leqslant 39. \end{array}$$

The reader may verify easily and quickly that these values, 3, 5, 11, 17 and 41 are the only values of A less than 100 which yield a similar run of primes and he is warned that Dickson's *History* records that Escott, in 1910, examined values of A " much exceeding 54000 " without finding a suitable A greater than 41.

2. The best known of the expressions in the previous section is x^2+x+41. There are in the range quoted 80 successive values of x for which the expression gives a prime number : the primes in order are

1601, 1523, 1447, 1373, ..., 43, 41, 41, 43, ..., 1373, 1447, 1523, 1601.

According to Dickson it was stated by Miot, in 1912, that

$$x^2-2999x+2248541 \text{ is prime for } 1460 \leqslant x \leqslant 1539.$$

This expression is obtained from x^2+x+41 by replacing x by $x-1500$ and the primes it yields for the successive values of x are the same 80 in the same order as quoted above.

But Miot's expression is only one of an infinite number of expressions derived from x^2+x+41 which yield for a range of 80 successive values of x the same 80 primes as those quoted and in the same order. The general result, obtained by substituting $x+n$ for x in x^2+x+41 is that, n being an integer.

$$x^2+(2n+1)x+(n^2+n+41) \text{ is a prime for } -(n+40) \leqslant x \leqslant -(n-39).$$

As special expressions we may mention

$$\begin{array}{lll} x^2+x+41, & \text{prime for} & -40 \leqslant x \leqslant 39 \\ x^2-79x+1601, & \text{prime for} & 0 \leqslant x \leqslant 79 \\ x^2-x\quad+41\quad, & \text{prime for} & -39 \leqslant x \leqslant 40 \\ x^2+79x+1601, & \text{prime for} & -79 \leqslant x \leqslant 0 \end{array}$$

The first was noted by Escott in 1899 : Legendre found that the first is prime for $0 \leqslant x \leqslant 39$ and Euler that it is prime for $0 \leqslant x \leqslant 15$. Euler also noted that the third is prime for $1 \leqslant x \leqslant 40$.

3. The following general results are deduced similarly from the other expressions in section 1 above, taken in order, and similar remarks as to the number and the run of primes apply :

(i) $x^2+(2n+1)x+(n^2+n+1)$ is prime for $-(n+4) \leqslant x \leqslant -(n-3)$
(ii) $x^2+(2n+1)x+(n^2+n+3)$ is prime for $-(n+2) \leqslant x \leqslant -(n-1)$
(iii) $x^2+(2n+1)x+(n^2+n+5)$ is prime for $-(n+4) \leqslant x \leqslant -(n-3)$
(iv) $x^2+(2n+1)x+(n^2+n+11)$ is prime for $-(n+10) \leqslant x \leqslant -(n-9)$
(v) $x^2+(2n+1)x+(n^2+n+17)$ is prime for $-(n+16) \leqslant x \leqslant -(n-15)$

4. Next let us consider the expression $2x^2+A$. It is found that

$2x^2+5$ is prime for $-4 \leqslant x \leqslant 4$
$2x^2+11$ is prime for $-10 \leqslant x \leqslant 10$
$2x^2+29$ is prime for $-28 \leqslant x \leqslant 28$

and the values 5, 11, 29 appear to be the only suitable values of A less than 500.

If, in these three expressions taken in order, we substitute $x+n$ for x we have the corresponding general results :

(i) $2x^2+4nx+(2n^2+5)$ is prime for $-(n+4) \leqslant x \leqslant -(n-4)$
(ii) $2x^2+4nx+(2n^2+11)$ is prime for $-(n+10) \leqslant x \leqslant -(n-10)$
(iii) $2x^2+4nx+(2n^2+29)$ is prime for $-(n+28) \leqslant x \leqslant -(n-28)$

and in each of the three cases the primes are the same, and occur in the same order, as when $n=0$.

5. The expressions $1+Ax-x^2$ and x^2-Ax-1 are also of interest. For certain values of A the first yields a run of primes for $0 \leqslant x \leqslant A$ and the second gives a run for $A+1 \leqslant x \leqslant 2(A-1)$. For example

(i) $1+5x-x^2$ is prime for $0 \leqslant x \leqslant 5$ and x^2-5x-1 for $6 \leqslant x \leqslant 8$
(ii) $1+7x-x^2$ is prime for $0 \leqslant x \leqslant 7$ and x^2-7x-1 for $8 \leqslant x \leqslant 12$
(iii) $1+13x-x^2$ is prime for $0 \leqslant x \leqslant 13$ and $x^2-13x-1$ for $14 \leqslant x \leqslant 24$
(iv) $1+17x-x^2$ is prime for $0 \leqslant x \leqslant 17$ and $x^2-17x-1$ for $18 \leqslant x \leqslant 32$

The values 5, 7, 13, 17 are the only suitable values of A less than 100. The third pair was mentioned by Mr. Boon in the Note referred to above. If, as before, we replace x by $x+n$ in each of these expressions, we find

from (i) that

$(1+5n-n^2)-(2n-5)x-x^2$ is prime for $-n \leqslant x \leqslant -(n-5)$
$x^2+(2n-5)x+(n^2-5n-1)$ is prime for $-(n-6) \leqslant x \leqslant -(n-8)$

from (ii) that

$(1+7n-n^2)-(2n-7)x-x^2$ is prime for $-n \leqslant x \leqslant -(n-7)$
$x^2+(2n-7)x+(n^2-7n-1)$ is prime for $-(n-8) \leqslant x \leqslant -(n-12)$

from (iii) that

$(1+13n-n^2)-(2n-13)x-x^2$ is prime for $-n \leqslant x \leqslant -(n-13)$
$x^2+(2n-13)x+(n^2-13n-1)$ is prime for $-(n-14) \leqslant x \leqslant -(n-24)$

from (iv) that

$(1+17n-n^2)-(2n-17)x-x^2$ is prime for $-n \leqslant x \leqslant -(n-17)$
$x^2+(2n-17)+(n^2-17n-1)$ is prime for $-(n-18) \leqslant x \leqslant -(n-32)$

and, again, in each of the four cases the primes are the same, and occur in the same order as when $n=0$.

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2611. *The integration of* $\sec^3 x$ *etc.*

The integration of $\sec^3 x$ and allied integrations—including that of $\sqrt{(a^2+x^2)}$ —can be effected in various well-known ways. One way uses the standard reduction formulae for integration of powers of $\cos x$ or $\sin x$ applied to negative powers. The following essentially equivalent method does not presuppose a knowledge of these formulae, which in fact it gives incidentally (and without needing integration by parts).

Let D denote d/dx (with D^2 and D^{-1} for d^2/dx^2 and $\int dx$); let n be integral; and let z_n denote either $\cos^n x$ or $\sin^n x$.

Differentiation twice followed by one integration gives

$$D^2 z_n = n(n-1)z_{n-2} - n^2 z_n, \quad n(n-1)D^{-1}z_{n-2} = (D+n^2D^{-1})z_n.$$

Taking $n=-1$, -2, etc. and using the known integrals of $\sec x$, $\sec^2 x$, $\operatorname{cosec} x$, $\operatorname{cosec}^2 x$ we can write down the integrals of other powers of $\sec x$, $\operatorname{cosec} x$. Thus

$$\int \sec^3 x\,dx = \tfrac{1}{2}(D+D^{-1})\sec x = \tfrac{1}{2}[\sec x \tan x + \log(\sec x + \tan x)],$$

$$\int \sec^4 x\,dx = \tfrac{1}{6}(D+4D^{-1})\sec^2 x = \tfrac{1}{3}(\sec^3 x \tan x + 2\tan x),$$

$$\int \operatorname{cosec}^5 x\,dx = \tfrac{1}{12}(D+9D^{-1})\operatorname{cosec}^3 x = -\tfrac{1}{4}\operatorname{cosec}^3 x \cot x + \tfrac{3}{8}(D+D^{-1})\operatorname{cosec} x$$
$$= -\tfrac{1}{4}\operatorname{cosec}^3 x \cot x - \tfrac{3}{8}\operatorname{cosec} x \cot x + \tfrac{3}{8}\log\tan\tfrac{1}{2}x,$$

or equivalent forms.

C. W.

2612. *Definitions of e and* π.

In Note 1805 (February 1945) Mr. C. O. Tuckey advocated the visual appeal to the graphs of a^x for different values of a to make evident the existence of a number e for which the slope of e^x at (0, 1) is 1. This leads immediately to $De^x=e^x$. Similar ideas applied to the graphs of $\sin x$ for different units of angle reveal the existence of a unit of angle for which the slope of $\sin x$ at (0, 0) is 1; and this leads immediately to $D\cos x = -\sin x$ and $D\sin x = \cos x$ with this choice of unit. (See Note 1956, May 1947.)

While there can be little doubt as to the suitability of this approach for some types of students (e.g. prospective engineers) it undoubtedly leaves something to be desired from the strictly logical viewpoint, and for that reason other methods are generally preferred. These other methods (for the exponential) almost invariably break away from familiar ideas of "indices" and avoid any attempt to define a^x, which remains undefined except for rational values of x until later. To some minds this appears to shirk the issue without even so much as looking at the problem to see what it is. The question is, cannot the theory be developed logically from this direct viewpoint?

The direct definition of a^x for positive real values of a and irrational values of x (and most rational values) must depend on the definition used for irrational numbers, That means, in some form, the Dedekind definition or an equivalent. (A definition of a^x on these lines is given in my *Mathematical Analysis.*) In what follows such a definition is presupposed.

Using the notation $(x \mid y)$ to denote the Dedekind classification of the real numbers into the x and y classes, the number e can be defined thus:

Def. 1. $e=(b \mid B)$ *such that b denotes any number* $\leqslant 1$ *or any number* >1 *for which* $(b^h-1)/h \leqslant 1$ *for some* $h>0$ *and B denotes any number* >1 *for which* $(B^h-1)/h>1$ *for all* $h>0$.

The three conditions needed for the validity of this definition are: (i) every real number is classified, (ii) every $b <$ every B, (iii) both classes (b) and (B) exist.

Conditions (i), (ii) are almost automatic, (i) by direct logic, (ii) because x^h increases with x for $h>0$ and $x>0$. Condition (iii) is a consequence of the "convexity" (See Hardy, Littlewood and Polya, *Inequalities*, Ch. III) of a^x

since the choice $B=4$ with $h>0>-\frac{1}{2}=H$ entails $(B^h-1)/h>(B^H-1)/H=1$, so that class (B) exists; the existence of (b) is obvious. (The convexity of a^x follows from

$$a^{x-h}+a^{x+h}-2a^x=a^{x-h}(1-a^h)^2>0.)$$

The crucial limit $(e^h-1)/h\to 1$ as $h\to 0$ follows from this definition, and with it, $De^x=e^x$.

For the trigonometric functions the definitions of $\cos x$, $\sin x$, etc., needed are the elementary definitions for acute angles,—whereby such facts as $\sin(\frac{1}{3}$ right angle$)=\frac{1}{2}$, $0<\sin x<\tan x$, etc., are known. Using the right angle as the unit of angle the definition for π is:

Def. 2. *$\pi=(p\mid P)$ such that p denotes any number $\leqslant 0$ or any positive number for which $\dfrac{1}{h}\sin\dfrac{2h}{p}\geqslant 1$ for some $h>0$ and P denotes any positive number for which $\dfrac{1}{h}\sin\dfrac{2h}{P}<1$ for all $h>0$, where the angles are measured in right angles.*

Again the first two of the three Dedekind conditions are almost automatic ((ii) because $\sin x$ increases with x). Condition (iii) follows from the concavity of $\sin x$ and convexity of $\tan x$ (and $\sin x<\tan x$) for $0<x<1$, since the choice $P=4$ with $0<h<1$ gives $\dfrac{1}{h}\sin\dfrac{2h}{P}<\dfrac{1}{h}\tan\dfrac{h}{2}<\tan\dfrac{1}{2}=1$, so that $\dfrac{1}{h}\sin\dfrac{2h}{P}<1$ for $0<h<1$ and therefore also for all positive h. (The concavity of $\sin x$ follows from $\sin(x-h)+\sin(x+h)=2\sin x\cos h<2\sin x$ and the convexity of $\tan x$ from

$$\tan(x-h)+\tan(x+h)=2\tan x(1+\tan^2 h)/(1-\tan^2 x\tan^2 h)>2\tan x$$

for $0<x-h<x+h<1$. It may be noticed that the choice $p=3$ with $h=\frac{1}{2}$ would make $\dfrac{1}{h}\sin\dfrac{2h}{p}=2\sin\dfrac{1}{3}=1$, so that $3<\pi<4$.)

The crucial limit $\left(\sin\dfrac{2h}{\pi}\right)\Big/h\to 1$ as $h\to 0$ follows, and with it and the easy limit $\left(\cos\dfrac{2h}{\pi}-1\right)\Big/h\to 0$, the results

$$D_x\cos\frac{2x}{\pi}=-\sin\frac{2x}{\pi},\quad D_x\sin\frac{2x}{\pi}=\cos\frac{2x}{\pi},$$

with the angles measured in right angles; which are the same as

$$D\cos x=-\sin x,\quad D\sin x=\cos x$$

with the angles measured in such units ("radians") as make $\frac{1}{2}\pi$ radians $=1$ right angle.

C. W.

2613. *On Notes* **2466** *and* **2338.**

Note 2466 gives a proof by pure geometry that the intercept on the tangent of the three-cusped hypocycloid is of constant length. This is easily deduced from the following two lemmas, which may be proved quite simply by pure methods.

LEMMA 1: If a circle radius a rolls in a circle radius $a+b$, where a and b are mutually prime integers, a equally spaced points on it generate a hypocycloid with $a+b$ cusps. A second generation of the curve is provided by b points on a rolling circle of radius b.

LEMMA 2: The same hypocycloid is enveloped by a and b equally spaced

diameters on circles of radii $2a$ and $2b$, respectively, rolling in the same manner (with the qualification that one of the motions is now pericyclic).

To prove the required result put $a=1$ and $b=2$ and use both lemmas. It follows at once that a three-cusped hypocycloid is generated in three ways simultaneously by rolling a circle radius 2 in a circle radius 3. Two points on the rolling circle generate the curve while the diameter joining them envelopes it.

We have not used the second tangential generation of the curve. This shows that the curve is enveloped by two perpendicular diameters of a circle radius 4 rolling pericyclically round a circle radius 3. It follows that two of the three tangents which can be drawn to the curve from points on the inscribed circle are perpendicular; that is to say the inscribed circle is part, at least, of the orthoptic locus. In fact it is the whole of it.

Note 2338 and earlier notes proved that the envelope of the Simson line is a three-cusped hypocycloid. The interest is again in giving a simple proof without using calculus. This is easily done using two further results which are proved in Maxwell's *Geometry for Advanced Pupils* pp. 48–50 and in other similar books.

LEMMA 3: The Simson line of P, a point on the circumcircle of the triangle ABC with orthocentre H, bisects PH at a point Q on the nine-point circle.

The last part of this is not stated explicitly in the reference, but it follows immediately since Q and P generate homothetic figures with H as the homothetic centre. Also if P is displaced round the circumcircle by an angle θ then Q is displaced round the nine-point circle by the same amount.

LEMMA 4: If the perpendicular from P to BC meets the circumcircle again at L', then the Simson line is parallel to AL'.

It follows from this that if P is given a displacement θ round the circumcircle then the Simson line rotates by an amount $-\theta/2$. (This is because the angle at the circumference is half the angle at the centre.)

Combining these two results we see that the motion of the Simson line can be produced by moving Q round the nine-point circle with angular velocity $\dot{\theta}$ whilst rotating a line through it with angular velocity $-\dot{\theta}/2$. But this is just the motion of the diameter of a circle radius 2 rolling in a circle radius 3, and we have seen already that this envelops a three-cusped hypocycloid.

Many of these properties are illustrated in my films "The Simson Line" and "The Cardioid".

T. J. FLETCHER

2614. *The Harmonic and the Polar Transformations.*

1. A projective relation.

(*a*) The operations used in the construction of the harmonic polar p of a given point P for a triangle (U) are projective. The same is true of the converse operation of determining the harmonic pole P of a given line p [1; pp. 244, 245]. This may be stated succinctly: *The harmonic relation of a point and a line for a triangle is projective.*

(*b*) Thus if a figure consisting of the three elements P, p, (U) (§ 1*a*) is subjected to a projective transformation, the elements of the resulting figure will have to each other the same relation as the given elements P, p, (U).

In particular, if we transform P, p, (U), by the polarity with respect to a conic (K), we obtain the pole P' of p, the polar p' of P, and the polar reciprocal triangle (U') of (U). Hence the

THEOREM. *If a point P and a line p are harmonic for a triangle (U), and (U') is the polar reciprocal of (U) for a conic (K), the pole and polar of p, P for (K) are harmonic for (U').*

(*c*) The Lemoine point N and the Lemoine axis n of a triangle (U) are harmonic for (U) and they are also pole and polar for the circumcircle (O) of (U), hence, by the preceding theorem, N, n are harmonic with respect to the polar reciprocal of (U) for (O), that is, for the tangential triangle (U') of (U). This is actually the case, since (U) is inscribed in (U'), and N, n are the centre and axis of perspectivity of the two triangles [1, pp. 260 ff.]

(*d*) THEOREM. *The centre of a central conic (K) and the polar, for (K), of the centroid of a triangle (U), are harmonic for the triangle (U') polar reciprocal of (U) for (K).*

Indeed, the trilinear polar, for (U), of the centroid G of (U) is the line at infinity i. Now the pole of i for (K) is the centre O of (K), hence O and the polar g, for (K), of G are harmonic for (U') (§ 1*b*).

The reader may consider the case when (K) is a parabola.

2. Converse proposition.

(*a*) THEOREM. *If X, x are pole and polar for a conic (K), and (U), (U') are two triangles polar reciprocal for (K), the trilinear pole of x for (U) and the trilinear polar of X for (U') are pole and polar with respect to (K).*

Let I be the harmonic pole of x for (U) and j the harmonic polar of X for (U'). The polarity for (K) transforms x, (U), I respectively into X, (U'), and the polar m of I for (K). On the other hand, since x, I are harmonic for (U), then X, m are harmonic for (U'), by the direct theorem (§ 1*b*). But the point X has only one harmonic polar for (U'), namely the line j, by construction, hence m and j coincide, and j is thus both the harmonic line of X for (U') and the polar line of I for the conic (K), which proves the proposition.

(*b*) Let us take for X, x (§ 2*a*) a focus and the corresponding directrix of (K), and for (U) any triangle inscribed in (K). The triangle (U') coincides with the tangential triangle of (U) for (K), and we have the proposition (§ 2*a*) : *The trilinear polar of the focus of a conic (K) for a triangle (U) inscribed in (K), and the trilinear pole, for the tangential triangle of (U) for (K), of the directrix corresponding to the focus considered, are pole and polar with respect to (K).*

(*c*) THEOREM. *If two triangles (U), (U') are polar reciprocal with respect to a central conic (K), the centroid of (U) and the harmonic polar, for (U'), of the centre of (K), are pole and polar with respect to (K).*

Indeed, the line at infinity i is the polar, for (K), of the centre of (K), and the harmonic pole of i for (U) is the centroid of (U), hence the proposition (§ 2*a*).

(*d*) In the special case when (U') (§ 2*c*) is inscribed in (K) we have the proposition : *The trilinear polar of the centre of a central conic (K), for a triangle (U') inscribed in (K), has for its pole, for (K), the centroid of the tangential triangle of (U') for (K).*

(*e*) As a special case of the preceding proposition (§ 2*d*) we have : *The polar of the centroid of a triangle with respect to a tritangent circle coincides with the harmonic polar of the centre of the circle considered with respect to the triangle formed by the points of contact of that circle with the sides of the given triangle.*

3. The self-polar triangle.

If the triangle (U) (§§ 1*b*, 2*a*) is polar with respect to the conic (K), those two propositions become, respectively,

(*a*) THEOREM. *If X, x are harmonic with respect to a triangle (U) polar with respect to a conic (K), the pole of x and the polar of X, for the conic (K), are also harmonic with respect to (U).*

(*b*) Converse Theorem. *If X, x are pole and polar for a conic (K), and (U) is a polar triangle for (K), the harmonic pole of x and the harmonic polar of X, for the triangle (U), are also pole and polar for (K).*

4. Applications.

(*a*) Let M be a point on a central conic (K), and let m be the symmetric, with respect to the centre O of (K), of the line joining the projections of M upon the axes a, b of (K).

Let t be the tangent to (K) at M, and let T be the symmetric, with respect to O, of the fourth vertex of the rectangle whose three other vertices are O, ta, tb.

THEOREM. *The point T and the line m are pole and polar for the conic (K).*

Indeed, the triangle abi formed by the lines a, b and the line at infinity i is self-polar for (K). Now, by construction, M, m are harmonic with respect to the triangle abi, and the same holds for T, t, hence the proposition (§ 3*b*).

(*b*) The propositions §§ 1*d*, 2*c*, in the case the triangle (U) is polar for the central conic (K), may be stated as follows: *If (U) is a triangle polar for a central conic (K), (i) The polar of the centroid of (U) for (K) and the centre of (K) are harmonic for (U). (ii) The centroid of (U) and the harmonic polar, for (U), of the centre of (K), are pole and polar with respect to the conic.*

The special case of those two propositions when the conic (K) coincides with the polar circle of the triangle (U) has been called attention to recently by John Leech (this *Gazette*, Vol. 38, No. 324, May 1954, p. 118, note 2397).

5. Generalizations.

(*a*) THEOREM. *If a point L and a plane λ are harmonic with respect to a tetrahedron (T), and (T') is the polar reciprocal tetrahedron of (T) for a quadric (Q), the polar plane λ' of L and the pole L' of λ for (Q), are harmonic with respect to (T')* [2].

(*b*) CONVERSE THEOREM. *If M, μ are pole and polar plane for a quadric (Q), and (T), (T') are two tetrahedrons polar reciprocal for (Q), then the harmonic pole of μ for (T) and the harmonic plane of M for (T') are pole and polar plane with respect to the quadric (Q).*

The two propositions are generalizations for three dimensional space of the two propositions §§ 1*b*, 2*a*. The proofs given for the case of the plane are applicable, without modification, to the propositions in space.

Presumably those propositions are valid in Euclidean spaces of higher dimensions.

The developments given above in the plane may readily be duplicated in space.

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2615. *A Combinatorial Identity.*

Höhn [2] has shown that the quantities s_k which denote angle-sums in an n-dimensional simplex satisfy the equations

$$\sum_{\beta=0}^{k} (-1)^{\beta} \binom{n+1-\beta}{n+1-k} s_{\beta} = s_k \qquad \text{.........(1)}$$

where $k=0, 1, \ldots, n+1$. An independent subset of this set of equations is obtained by taking alternate equations (including the last equation). Sommerville [3] had earlier given the equations

$$\sum_{k=r}^{n+1} (-1)^k \binom{k}{r} s_k = \sum_{k=n+1-r}^{n+1} (-1)^{n+1-k} \binom{k}{n+1-k} s_k \qquad \text{.........(2)}$$

where $r=0, 1, \ldots, [\frac{1}{2}n]$. Coxeter [1] has remarked that the equivalence of (1) and (2) is not immediately obvious; it is the purpose of the present note to prove this equivalence.

As usual, we shall use $\binom{n}{r}$ to denote the coefficient of t^r in the expansion of $(1+t)^n$; it follows that

$$\binom{n}{r}=0 \quad \text{for integers } n<r, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$\binom{-a}{k}=(-1)^k\binom{a+k-1}{k} \quad \text{for } a>0. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

Our proof will, in addition, require the following

LEMMA

$$\sum_{k=r}^{n+1}(-1)^k\binom{k}{r}\binom{n+1-\beta}{n+1-k}=(-1)^{n+1}\binom{\beta}{n+1-r}.$$

Proof. Compare, using formulae (3) and (4), the coefficients of t^{n+1-r} on both sides of the identity

$$(1+t)^{-r-1}(1+t)^{n+1-\beta}=(1+t)^{n-r-\beta}.$$

The lemma follows at once.

We may now deduce equations (2) from equations (1); we find

$$\sum_{k=r}^{n+1}(-1)^k\binom{k}{r}s_k=\sum_{k=r}^{n+1}(-1)^k\binom{k}{r}\sum_{\beta=0}^{k}(-1)^\beta\binom{n+1-\beta}{n+1-k}.\ s_\beta$$

Formula (3) allows us to extend the range of summation on β to the whole interval $(0, n+1)$; the lemma just proved then gives the result

$$\sum_{\beta=0}^{n+1}(-1)^{n+1-\beta}\binom{\beta}{n+1-r}s_\beta.$$

The only non-zero binomial coefficients are those with $\beta \geqslant n+1-r$; we thus end up with the Sommerville equations (2).

Conversely, putting $r=0$ in (2), gives

$$s_{n+1}=\sum_{k=0}^{n+1}(-1)^k s_k$$

which is one of the Höhn equations (1). We proceed by induction, assuming the equations (1) do hold for all values greater than a certain value r. Then

$$\sum_{k=r+1}^{n+1}(-1)^k\binom{k}{r}s_k+(-1)^r s_r$$
$$=\sum_{k=r+1}^{n+1}(-1)^k\binom{k}{r}\sum_{\beta=0}^{k}(-1)^\beta\binom{n+1-\beta}{n+1-k}s_\beta+(-1)^r s_r.$$

This expression is also equal to

$$\sum_{\beta=n-r+1}^{n+1}(-1)^{n-\beta+1}\binom{\beta}{n-r+1}s_\beta=\sum_{k=r}^{n+1}(-1)^k\binom{k}{r}\sum_{\beta=0}^{n+1}(-1)^\beta\binom{n+1-\beta}{n+1-k}s_\beta$$

and so we obtain, by comparsion,

$$s_r=\sum_{\beta=0}^{r}(-1)^\beta\binom{n+1-\beta}{n+1-r}s_\beta$$

which is the Höhn equation for s_r; we have thus established (1) from (2) by a backwards induction.

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1. H. S. M. Coxeter, *Math. Rev.*, 15, 1 (1954), p. 55.

2. W. Höhn, Thesis, Zurich, 1953.

3. D. M. Y. Sommerville, *Proc. Roy. Soc.*, London Ser. A 115 (1927), pp. 103–119.

D. A. Sprott

2616. *On Note* 1941.

Mr. B. A. Swinden's "solution" from a script has greater generality than appears at first sight. For convenience, the "solution" is repeated (my language):

Given a triangle ABC in which $a=4$, $b=7$, $c=9$, to calculate the angle C.

Probably beginning with a rough sketch in which the angle at B happened to be nearly a right angle, the candidate assumed the formula

$$\sin C = c/b,$$

and was inspired to continue

$$\begin{aligned}\sin C &= 9/7\\ &= 1{\cdot}2857\\ &= 1 + {\cdot}2857\\ &= \sin 90^\circ + \sin 16^\circ\, 36'\\ &= \sin 106^\circ\, 36',\end{aligned}$$

so that, correctly,

$$C = 106^\circ\, 36'.$$

The purpose of this note is to give a geometrical construction for triangles that can be solved by this technique. [Note that inverted commas are not now necessary round the word *solved*, as the method can be proved to lead to the correct answer.]

Let OBC be a triangle right-angled at C. Draw the circle with centre B and radius $BO+BC$ to cut the circle with centre C and radius $CO+CB$ in A. Then ABC is a triangle of required type.

E. A. Maxwell

2617. *Magic matrices.*

In their note on Magic Squares (*Mathematical Gazette*, Vol. XXXIX (1955), p. 132 note number 2505) Messrs. A. D. Booth and K. H. V. Booth define a magic square as one whose rows and columns each have the same sum. Some definitions of magic squares require more than this, however. Not only must the terms in each row and in each column add up to the same number, s say, but the terms of each diagonal must also add up to s. This is the definition given by W. W. R. Ball in his "Mathematical Recreations and Problems" and by Hermann Shubert in his "Mathematical Essays and Recreations" (Open Court Publishing Company).

Using this, the more common definition, as a basis we shall say that the $n\times n$ square matrix M, with elements $m_{i,j}$ $(i=1, 2, \dots n)(j=1, 2, \dots n)$, is a magic matrix if

$$(1)\quad \sum_{i=1}^{n} m_{i,j}\ (j=1, 2 \dots n) = \sum_{j=1}^{n} m_{i,j}\ (i=1, 2, \dots n) = \sum_{i=1}^{n} m_{i,i} = \sum_{i=1}^{n} m_{i,n+1-i} = s.$$

The last two sums in (1) are taken along the diagonals of M. I shall not require the elements $m_{i,j}$ to be integers.

Using (1) as our definition I shall prove that the inverse of a 3×3 magic matrix M, with row, column and diagonal sum s, is a magic matrix with row, column and diagonal sum $1/s$.

Before giving the proof I give an example. The matrix inverse to M is denoted, as usual, by M^{-1}.

$$M = \begin{pmatrix} 2 & 7 & 6 \\ 9 & 5 & 1 \\ 4 & 3 & 8 \end{pmatrix}, \qquad M^{-1} = \begin{pmatrix} -37/360 & 38/360 & 23/360 \\ 68/360 & 8/360 & -52/360 \\ -7/360 & -22/360 & 53/360 \end{pmatrix}.$$

To prove the result just stated we need only discuss the diagonal terms. For in Messrs. Booth's note cited above the proof that each row and column sum of M^{-1} is $1/s$, is independent of any condition along the diagonals and so is applicable here. But as they do not discuss the diagonal sums it is necessary for us to do so.

We first deal with the principal diagonal, i.e. the diagonal which runs from the top left-hand corner to the bottom right-hand corner. We need to discuss the characteristic equation of M, i.e. the cubic equation in x given by

$$(2) \qquad \begin{vmatrix} m_{1,1} - x & m_{1,2} & m_{1,3} \\ m_{2,1} & m_{2,2} - x & m_{2,3} \\ m_{3,1} & m_{3,2} & m_{3,3} - x \end{vmatrix} = 0$$

We shall denote the roots of (2) by x_1, x_2 and x_3.

On adding along the columns and using (1), (2) can be written in the form

$$(3) \qquad (s-x) \begin{vmatrix} 1 & 1 & 1 \\ m_{2,1} & m_{2,2} - x & m_{2,3} \\ m_{3,1} & m_{3,2} & m_{3,3} - x \end{vmatrix} = 0.$$

Now the coefficient of x in the determinant is $m_{2,1} + m_{3,1} - m_{2,2} - m_{3,3} = (m_{1,1} + m_{2,1} + m_{3,1}) - (m_{1,1} + m_{2,2} + m_{3,3}) = s - s = 0$, by (1). Hence the characteristic equation takes the form

$$(4) \qquad (s-x)(x^2+K) = Ks - Kx + sx^2 - x^3 = 0,$$

where K is a constant independent of x. Thus finally we deduce from (4), by means of the theory of algebraical equations, that

$$(5) \qquad \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = \frac{1}{s}.$$

To apply this equation we note the following facts, true for all $n \times n$ matrices:

(i) the characteristic roots of M^{-1} are the reciprocals of the characteristic roots of M,

(ii) the sum of the elements of the principal diagonal of a matrix = the sum of the characteristic roots.

Hence, in our case with $n = 3$, it follows that the elements of the principal diagonal of M^{-1} have a sum equal to

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = \frac{1}{s},$$

by (5), and so equal to the sum of any row or column of M^{-1}.

To prove a like result for the secondary diagonal, i.e. the diagonal of M^{-1} which runs from the bottom left-hand corner to the top right-hand corner, we make use of the matrix J defined as follows :

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{6}$$

It is easily shown that $JJ = I$ (the unity matrix) and therefore that MJ and JM^{-1} are inverse matrices.

Now the elements of MJ are those of M reflected in the middle column of M and so MJ is a magic matrix with row, column and diagonal sum equal to s. Hence, from the proof above, it follows that the principal diagonal of JM^{-1} has $1/s$ for the sum of its elments.

But the elements of JM^{-1} are those of M^{-1} reflected in the middle row of M^{-1} and so the sum along the secondary diagonal of M^{-1} is equal to the sum along the principal diagonal $JM^{-1} = 1/s$.

This completes the proof of our statement that the inverse of a 3×3 magic matrix with row, column and diagonal sum s is a magic matrix with row, column and diagonal sum $1/s$.

It would be interesting to know whether this result can be extended to $n \times n$ magic matrices for values of n other than $n = 3$. The proof above depends upon equation (5) which in turn depends upon showing that the coefficient of x in the determinant of (3) is equal to zero. When $n = 3$ this coefficient is linear in the elements $m_{i,j}$ and so is easily dealt with. When $n \neq 3$ this coefficient is no longer linear in $m_{i,j}$ and is much more difficult to discuss.

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2618. *A property of Apollonius circles.*

If B, C are fixed points and λ a given ratio, the locus of a point P which moves so that $PB/PC = \lambda$ is well known to be a circle, with centre on BC produced, having B, C as inverse points and is commonly referred to as "Apollonius' Circle". Let this locus be named more specifically as the Apollonius circle of BC with ratio λ.

If ABC is a given triangle the Apollonius circles α, β, γ of BC, CA, AB with ratios AB/AC, BC/BA, CA/CB pass through A, B, C respectively and are well known to have two common points at either of which $PA.BC = PB.CA = PC.AB$. They may be called the Apollonius circles of the triangle ABC.

The object of this Note is to state and prove the further simple property that the Apollonius circles of any triangle intersect at angles of 60° or 120°.

Taking O, one of the common points of α, β, γ, as centre of inversion let the inverses of A, B, C, α, β, γ be A', B', C', α', β', γ'.

α passes through O and A and has B, C as inverse points. Hence α' is a straight line through A' and is the perpendicular bisector of $B'C'$. Thus $A'B' = A'C'$. Repeating this argument it appears that α', β', γ' are altitudes of an equilateral triangle (the circumcentre of which is the inverse of the remaining common point of α, β, γ) and the stated property is established.

Incidentally the two common points of the Apollonius circles of a triangle provide solutions to the problem: Determine a point with respect to which the vertices of a given triangle invert into those of an equilateral triangle.

T. A. HONAN

2619. *Tangent, chord theorem.*

This proof, for the acute angle, follows directly the "angle at centre" theorem instead of the dependent theorems, "angle in a semi-circle", "angles in same segment".

Using the notation of Note 2462,

$$
\begin{aligned}
E &= \tfrac{1}{2}O_1 && \text{(angle at centre)}\\
&= \tfrac{1}{2}(180^\circ - A_1 - D_1) && \text{(angle sum of } \triangle)\\
&= \tfrac{1}{2}(180^\circ - 2A_1) && (OA = OD, \text{ radii})\\
&= 90^\circ - A_1 &&\\
&= A_2. && \text{(Tangent, chord).}
\end{aligned}
$$

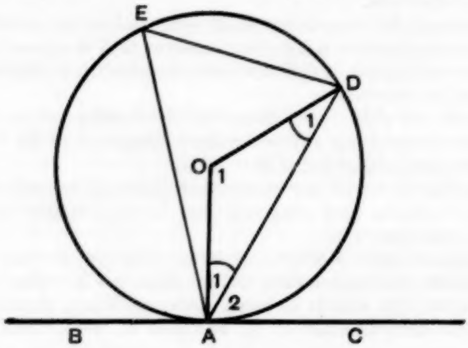

FIG. 1.

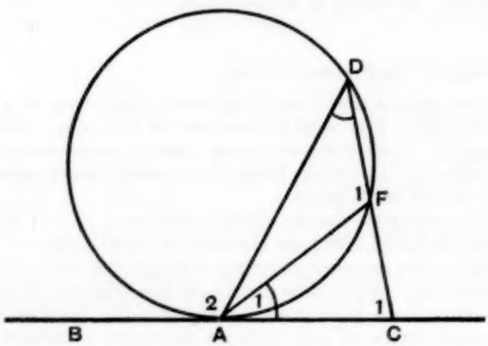

FIG. 2.

This proof, for the obtuse angle, avoids lines dividing the angles between the tangent and the chord.

$$
\begin{aligned}
F_1 &= A_1 + C_1 && (\text{Ext. } \angle \text{ of } \triangle.)\\
&= D + C_1 && \text{(Proved above)}\\
&= A_2 && (\text{Ext. } \angle \text{ of } \triangle)
\end{aligned}
$$

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2620. *Another pretty series.*

It may be worth adding the following to Mr. Ferguson's " pretty " series in Note 2419 (*Gazette*, May, 1954) :

$$\sum_{n=1}^{\infty} \binom{4n}{2n} x^{2n} = \tfrac{13}{7} \quad \text{when} \quad x = \tfrac{6}{25}.$$

D. G. TAHTA

2621. *On Note 2462 (Single letters for angles).*

The modified notation used below has the important advantage of indicating equal angles on the diagram ; this outweighs the disadvantage felt by some at the introduction of Greek letters, while retaining the original improvements.

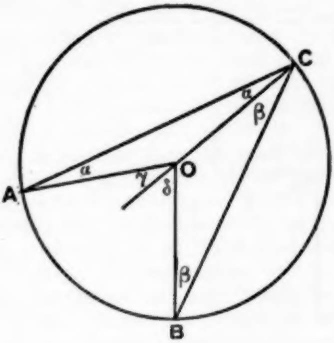


FIG. 1

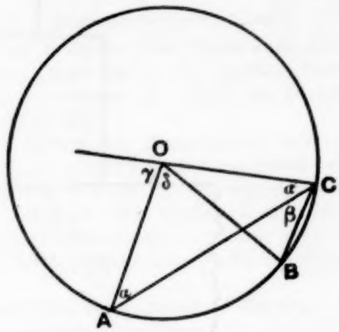


FIG. 2

In Fig. 1, $O_\gamma = A_\alpha + C_\alpha = 2C_\alpha$,

$O_\delta = 2C_\beta$,

and thus $O_{\gamma\delta} = 2C_{\alpha\beta}$.

In Fig. 2, $O_\gamma = 2C_\alpha$,

$O_{\gamma\delta} = 2C_{\alpha\beta}$,

and thus $O_\delta = 2C_\beta$.

The notation $C_{\alpha\beta}$ for the composite angle is clear and consistent while the relation

$$C_{\alpha\beta} = C_\alpha + C_\beta$$

has obvious analogies. A_α has been written rather than A for the sake of consistency.

If AD bisects the angle BAC, the halves of the angle may be marked α and α' and written A_α and $A_{\alpha'}$.

It will be noted that the method dispenses entirely with the three point notation—a further advantage.

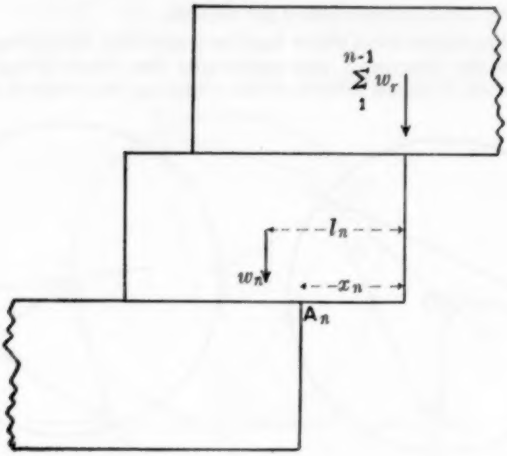
A. C. COSSINS

2622. *Statical absurdity.*

As an ever-welcome reminder to the incautious of the danger of intuitive thinking, I am setting out to prove a result which has been referred to by more than one experienced mathematician as " obviously absurd ".

Assuming uni-directional gravitation, it is possible, given sufficiently many homogeneous rectangular bricks, to build a stable column by placing them, one on another, irrespectively of size or weight, *in any order whatever*, in such a way that the top brick overhangs the bottom one by any specified distance, provided that there exist upper and lower bounds, W, w to the weights, and a lower bound l to the lengths of the bricks.

Consider a column in which the nth brick from the top (having weight w_n and length $2l_n$) overhangs the $(n+1)$th by $x_n (< l_n)$.



The whole column will be in limiting equilibrium if, taking moments about A_n for the forces acting on the nth brick,

$$w_n(l_n - x_n) = x_n \sum_1^{n-1} w_r, \quad \text{for all } n,$$

that is, if

$$w_n l_n = x_n \sum_1^{n} w_r, \quad \text{for all } n,$$

that is, if

$$x_n = w_n l_n \Big/ \sum_1^{n} w_r$$

(which is greater than or equal to $w_n l_n / nW$) for all n.

The overhang, O_{N+1}, for $N+1$ bricks is

$$\sum_1^N x_n \geqslant \sum_1^N (w_n l_n / nW)$$

$$\geqslant \frac{wl}{W} \sum_1^N \frac{1}{n}$$

which is divergent. Hence the overhang can be made to exceed any specified distance by taking N sufficiently large.

The displacement of the top brick in the appropriate direction by a finite amount $\epsilon\,(<l)$ will clearly render the whole column stable. In particular, if $W = w = l = 1$,

$$O_{N+1} = \sum_1^N \frac{1}{n} - \epsilon$$

$$= \log_e N + \gamma - \epsilon + \delta_{N+1},$$

(where γ is Euler's constant, and δ_N, which is positive, tends monotonely to zero as N tends to infinity).

Now, for $N > 10$, $\delta_N < \cdot 05$, and if $\nu = 10^6 N$, clearly

$$\delta_\nu - \delta_N < \cdot 05.$$

Thus $O_\nu - O_N = \log_e 10^6 + \delta_\nu - \delta_N < 14$ (slightly) for all $N > 10$.

Hence if more than 10 identical bricks are used, the effect of increasing their number one million-fold is to increase the overhang by slightly less than 14 units, or 7 brick lengths. There need be little fear, therefore, that the promulgation of this result will cause a revolution in architectural circles!

P. J. CLARKE

2623. *On Note* 2463 (*Maximum range of a projectile on any plane*).

Mr. Avery's conclusion may be extended to other than plane surfaces :

If a projectile is fired in a given vertical plane on to any surface, however irregular, the direction of motion, when it strikes the plane at maximum range, is at right angles to the direction of firing.

For if the parabolic envelope in the given vertical plane meets the surface at P, it is clear that no point of the surface beyond P in that plane can be reached ; and that the direction of motion at P must be the direction of the tangent to the envelope. To prove that this is at right angles to the direction of firing it is only necessary to differentiate the equation of the trajectory, first with respect to $\tan\theta$, and secondly with respect to x, and to eliminate gx/V^2 from the two results.

If O is the firing-point, it follows that OP is a focal chord of the trajectory-parabola. Hence the direction of firing to obtain maximum range may be found by bisecting the angle between OP and the vertical. E. H. L.

2624. *Tests for Divisibility for all numbers of the form* $10a\pm 1$ (*or their factors*).

Mr. Kashangaki's test for 19 (Note 2548) is most interesting, since, coupled with an extension to one for 7 given to me by Dr. D. S. Spence, it leads to a general test for divisibility by all numbers of the form $10a\pm 1$. Dr Spence's test for 7 (which he found as a boy) was to cut off the last figure and subtract twice this from what was left. Clearly this subtracts as many 21's as are needed to remove the last digit. To test for divisibility by $10a+1$, therefore, cut off the last digit and subtract a times the cut off figure from the rest. Thus, to test 987654321 for divisibility by 17, taking $a=5$, since $17\times 3=51$ we have in succession :

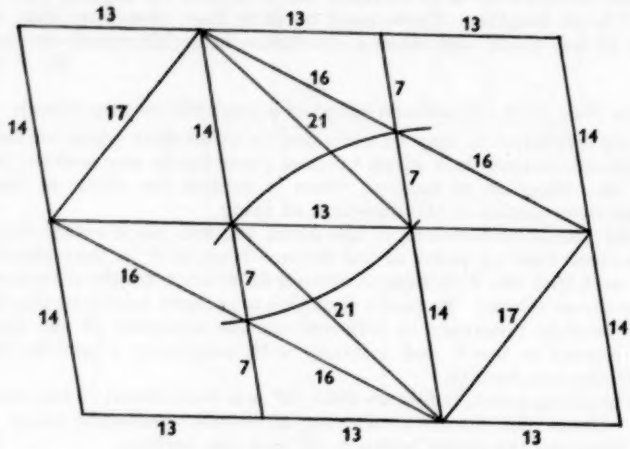
987654321
98765427
9876507
987615
98736
9843
969
51
0

Since this " goes exactly ", 987654321 is divisible in fact by 51. In such cases there is an interesting extension, for the quotient is given by the succession of crossed-off digits : viz. 19365771. The quotient on division by 17 is of course 3 times this : viz. 58097313. This method applies for 11, 21(7), 31, 41, 51(17), 61, 71, 81, 91(7 or 13), 101, 111(37), etc. Unfortunately it is only likely to be much used for 7 or 17, for which I think it is better than any yet found.

This led me to see at once that Mr. Kashangaki's method for 19 was also one of a general series of tests for $10a-1$, though this also covers much that most people would seldom need. However, since 23 (69/3) and 43 (129/3) are covered by it, we are now left with 47 as the smallest prime number for which there is not a pretty reasonable test for divisibility.

C. DUDLEY LANGFORD

2625. *Parallelograms with integral sides and diagonals.*



C. Dudley Langford

2626. *Schur's inequality.*

If $x+y+z=p$, then $(z-x)-(x-y)=p-3x$. Hence

$$\Sigma(p-3x)(x-y)(x-z)=(p-3x)(p-3y)(p-3z)$$

that is,

$$3f(x, y, z\,;\,1)=p\Sigma(x^2-yz)+2p^3-9p\Sigma yz+27xyz.$$

Also

$$p^2=\Sigma(x^2+2yz), \quad p\Sigma yz=9xyz+\Sigma x(y-z)^2$$

Hence

$$f(x, y, z\,;\,1)=p\Sigma(x^2-yz)-\Sigma x(y-z)^2$$

that is,

$$2f(x, y, z\,;\,1)=\Sigma x\,.\,\Sigma(y-z)^2-2\Sigma x(y-z)^2=\Sigma(y+z-x)(y-z)^2.$$

But

$$\begin{aligned}(x+y+z)(y+z-x)&=(y+z-x)^2+2x(y+z-x)\\&=(y+z-x)^2+2\{yz-(x-y)(x-z)\}\end{aligned}$$

and $\Sigma(x-y)(x-z)(y-z)^2=0$, identically. Hence we have

$$2(x+y+z)f(x, y, z\,;\,1)=\Sigma(y+z-x)^2(y-z)^2+2\Sigma yz(y-z)^2,$$

Watson's symmetrical identity.*

E. H. Neville

* On Schur's inequality. *Math. Gazette*; Vol. XXXIX, p. 207.

2627. *A generalisation of Schur's inequality.*

Schur's inequality, given by Hardy, Littlewood and Polya (*Inequalities*, Cambridge 1934, 64), proved by Barnard and Child (*Higher Algebra*, London 1936, 217) and discussed by Watson (*Math. Gaz.* 37(1953), 244–6 and 39(1955), 207–8), is a particular case of the following.

THEOREM. *Let $f(t)$ be a positive function of t, monotone or convex, in some interval (open or closed, finite, semi-infinite or infinite) and let x, y, z belong to this interval. Then, unless $x=y=z$,*

$$g \equiv f(x)(x-y)(x-z)+f(y)(y-z)(y-x)+f(z)(z-x)(z-y)>0.$$

PROOF. If $x=y\neq z$, we have $g=f(z)(z-x)^2>0$.
Hence we may suppose $x<y<z$, so that

$$0<f(y)\leqslant\max\{f(x),f(z)\}$$

by the monotony or convexity. Again

$$0<(z-y)(y-x)<(z-x)(y-x)=(x-y)(x-z)$$

and

$$0<(z-y)(y-x)<(z-y)(z-x).$$

Hence

$$f(y)(z-y)(y-x)<f(x)(x-y)(x-z)+f(z)(z-x)(z-y)$$

and $g>0$.

If we put $f(t)=t^u$ and take the interval to be the semi-infinite one in which $t>0$, we have Schur's inequality. The essential point, both of Schur's inequality and of the generalisation, is that only one of the three terms is negative and this negative term is always less in absolute value than the greater of the two positive terms. By a similar proof, if k is any odd positive integer and the conditions of the theorem are satisfied, we have

$$f(x)(x-y)^k(x-z)^k+f(y)(y-z)^h(y-z)^k+f(z)(z-x)^h(z-y)^k.$$

For even k, this result is, of course, trivial.

Of course, the conditions in the theorem are not necessary. If we suppose $x<y<z$ and write $x=y-h$, $z=y+k$, so that $h>0$ and $k>0$, (1) is equivalent to

$$f(y)<\left(1+\frac{h}{k}\right)f(y-h)+\left(1+\frac{k}{h}\right)f(y+k).$$

This certainly holds if

$$\max f(t)<4\min f(t)$$

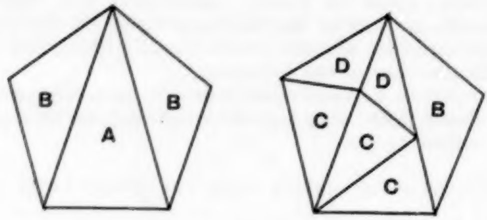
since

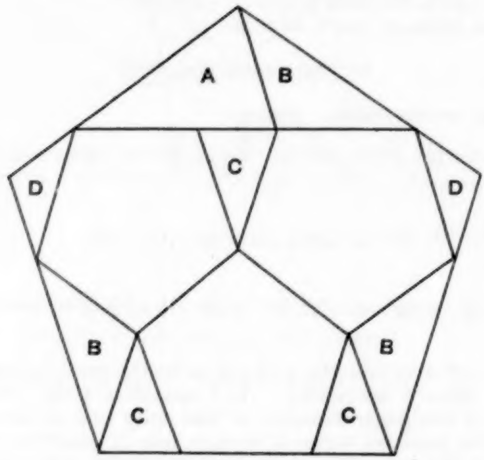
$$2+\frac{h}{k}+\frac{k}{h}\geqq 4.$$

Thus (1) holds over the whole real axis for such a function as $\dfrac{3+t^2}{1+t^2}$, which is neither monotone nor convex over this interval.

E. M. WRIGHT

2628. *To Pentasect a Pentagon.*





C. DUDLEY LANGFORD

2629. *Roots of a transcendental equation.*

Recently (*Math. Gaz.* 38(1954), 161–165) Goldenberg studied the equation

$$\coth z = (c/z) - b \qquad (b > 0) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and showed that, for $c < 1$, no root has a positive real part. I give here a shorter proof of this result and, in fact, prove a little more, viz.

THEOREM. *If c is real, the equation* (1) *has no non-real roots with positive real part. If* $c \leqslant 1$, *the equation has no positive real roots. If* $c > 1$, *the equation has just one positive real root.*

For $c < 0$, Goldenberg's proof is very simple and I do not repeat it. Next we suppose that $c > 0$, $z = x + iy$, $y \neq 0$ and $x > 0$. We separate (1) into real and imaginary parts and deduce that

$$\frac{\sinh 2x}{\sin 2y} = \frac{x - (b/c)(x^2 + y^2)}{y} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

and

$$\frac{\sin 2y}{\cosh 2x - \cos 2y} = \frac{cy}{x^2 + y^2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

two formulae given by Goldenberg. Our proof now diverges from his. From (3),

$$\frac{\sin 2y}{y}=\frac{c(\cosh 2x-\cos 2y)}{x^2+y^2}\geqslant\frac{c(\cosh 2x-1)}{x^2+y^2}>0$$

and, from (2),

$$\frac{y\sinh 2x}{\sin 2y}=x-\left(\frac{b}{c}\right)(x^2+y^2)<x.$$

Hence

$$1<\frac{\sinh 2x}{2x}<\frac{\sin 2y}{2y}<1,$$

a contradiction. Hence there are no roots with $y\neq 0$ and $x>0$; this is the first part of our theorem for $c>0$.

If $c>0$ and $y=0$, the equation (1) may be written as $\lambda(x)=0$, where

$$\lambda(x)\equiv\{(b+1)x-c\}e^{2x}-\{(b-1)x-c\}.$$

Now $\lambda((0)=0$ and

$$\begin{aligned}\lambda'(x)&=(b+1)e^{2x}+2\{(b+1)x-c\}e^{2x}-(b-1)\\&=(b-1)(e^{2x}-1)+2(b+1)xe^{2x}+2(1-c)e^{2x}\end{aligned}$$

and this is positive when $x>0$ and $c\leqslant 1$. Hence, for $c\leqslant 1$, we have $\lambda(x)>0$ and the equation has no positive real root.

If $c>1$, however, $\lambda'(0)=2(1-c)<0$ and

$$\begin{aligned}\lambda''(x)&=2(b-1)e^{2x}+2(b+1)e^{2x}+4(b+1)xe^{2x}+4(1-c)e^{2x}\\&=4e^{2x}\{b+1-c+(b+1)x\},\end{aligned}$$

so that $\lambda''(x)\gtreqless 0$ according as $x\gtreqless x_0=\max\left(0,\dfrac{c}{b+1}-1\right)$. Again $\lambda'(x)$ and $\lambda(x)$ both tend to $+\infty$ as $x\to+\infty$. It follows that, as x increases, $\lambda(x)$ decreases to a single minimum and thereafter steadily increases. Thus $\lambda(x)=0$ for just one positive x; that is, there is a single positive real root of (1).

Finally let $c=0$. Equation (1) is equivalent to

$$e^{2z}=\frac{b-1}{b+1}$$

and so

$$x=\tfrac{1}{2}\log\left|\frac{b-1}{b+1}\right|<1$$

for every root of (1). This completes the proof of our theorem.

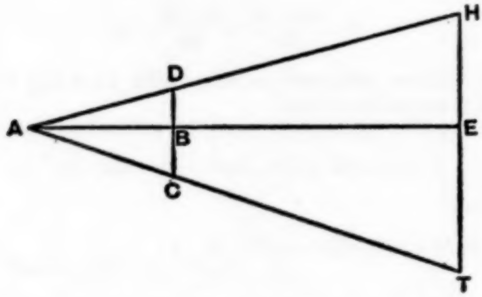
If we change the signs of b and z in (1), the equation is unaltered. Hence we can deduce from our theorem corresponding information for the case $b<0$.

E. M. WRIGHT

2630. *The slide rule used in measuring unknown distances.*

We have occasion to use facts about similar triangles. The point A is the observer's eye, CD the slide rule, and a the length of the outstretched arm. Holding the slide rule at B, we look up to see the point H, so that the ray AH

cuts the centimetre scale of the slide rule at the zero mark, D. Then we look down to see the point T, and note that the ray AT cuts the centimetre scale at C, giving a scale reading z. The height of the object TH (for example, a steeple) must be known or accurately estimated. If the object TH is horizontal, the angle TAH is in a horizontal plane. Now since the triangles ADC, AHT are similar, $AB/DC = AE/HT$; that is, $a/z = e/h$, or $e = ah/z$. If the length a of the arm is, for example, 60 cm., then $e = 60h/z$. This may be obtained by setting the divisor z directly above the dividend 6 on the slide rule, then shifting the hair-line of the runner on the factor h. We find e under the factor h.



$AB = a, \quad AE = e, \quad CD = z, \quad TH = h.$

The whole action in thus measuring unknown distances with the slide-rule comprises three steps:

(i) to look at the known object TH;
(ii) to locate the number z on the slide rule;
(iii) to locate the runner on the factor h.

The practical value of this method lies in

(i) the handiness of the slide rule (which may be a pocket slide rule);
(ii) the greater speed of application;
(iii) the obtaining of good estimates for the unknown distances.

This method is better than the " jacobsstab " (" thumb-jump " or " thumb-breadth ") method, because the length z on the centimetre scale is not constant, but suitable to the height, length or breadth of the known object TH.

Not only terrestrial measurements but also astronomical ones can be made. Looking at the full moon, we find $z = 6$ mm. Hence $e = 348000$ km., if the diameter of the moon is 3480 km. Looking at the sun through a sooty glass, we also find $z = 6$ mm. Hence the distance $e = 139000000$ km., if the diameter of the sun is 1390000 km. These distances are not far from the true values, and it is pleasant thus to measure the depths of the universe. Even if we do not want the distances of the moon and sun, it is very interesting to know that their distances from the earth are nearly one hundred times greater than their diameters.

There are many ways in which this method can be tested, for example in hiking; this will increase skill, and a control is available by comparison with the map.

F. Staber

REVIEWS

Projektive Differentialgeometrie. II. By G. BOL. Pp. v, 372. geb. DM 38. 1954. Studia Mathematica, 9. (Vandenhoeck & Ruprecht, Göttingen)

This second volume completes the study of Projective Differential Geometry which the author promised in the preface to the first volume. The first volume deals with curves and curve strips, while the second volume is devoted almost entirely to the theory of surfaces. For his study of surfaces the author introduces a technique which depends upon the use of asymptotic line parameters in terms of which the fundamental equations of surface theory take on a simple form. The consideration of the transformation from one set of parameters u and v to another set $u = f(u^*)$ $v = g(v)^*$ leads to a special type of transformation called a star transformation. A quantity D with a law of transformation of the form $D^* = f'^m g'^n D$ is said to be a semi-invariant of weight (m, n). The vanishing of a semi-invariant is therefore an invariant property for star formations and will have geometrical significance. With this object in view the author introduces a generalized differentiation such that the derivative of a semi-invariant is again a semi-invariant. This derivation is related also to the Cartan theory of alternating differential forms.

The present volume contains Chapters V–VIII of the main work. In Chapter V surface theory is developed in terms of the technique referred to. Among the geometrical illustrations of the formulae obtained, the theory of Lie quadrics is given. A definition of projective applicability is given which is a natural adaptation of metric applicability in classical differential geometry. It is also shown how a surface may be determined by the invariants occurring in its fundamental formulae.

In Chapter VII the generalized differentiation process introduced in Chapter V is adapted to differentiation along a curve, and applied to the treatment of hypergeodesics, pangeodesics, strips on a surface and conjugate nets.

In the final chapter the reference system is further simplified to what is known as the Wilczynski system and the advantages of the method for the treatment of certain geometrical problems is shown.

The book is most suitable for a University course in Projective Differential Geometry. It is particularly well supplied with examples illustrating the theory at every point, and also with references to original papers for the benefit of anyone wishing to study a particular topic in greater detail.

E. T. DAVIES

Plane Algebraic Curves. By E. J. F. PRIMROSE. Pp. vii, 111. 15s. 1955. (Macmillan)

This book is intended to give a reasonably brief introduction suitable for students who are unfamiliar with the theory of plane algebraic curves, and the author hopes that after reading the present book a student will be in a position to read more advanced books on the subject. It is intended for honours mathematics students at university, though the early part of the book could be studied by advanced sixth formers. Some knowledge of elementary geometry is assumed, together with the elements of homogeneous coordinates.

The following chapter headings give some idea of the contents: "Curves in the Real Euclidean Plane", "Rational Curves", "Line Equations", "Quadratic Transformations", "Intersection of two curves", "Plucker Equations", "Cubic Curves", "The Genus of a Curve".

The viewpoint adopted is perhaps too naive, at least for honours students, possibly due to the author's wish to make the book readable by weak students. For example, a plane algebraic curve is not even defined. Again on page 2 a

double point of a curve is defined to be a point P such that every line through P meets the curve twice at P. This appears to imply that a node is not a double point, since a tangent to a branch does not meet the curve twice. On the same page, an *asymptote* is defined to be a finite line which touches the curve at infinity. The term *finite line* is not defined, and since the chapter title is " *Curves in the Real Euclidean Plane* ", it might not be clear precisely what is meant. In this context a beginner may even interpret a finite line as one of finite length.

Usually it is fairly easy to see what the author really means, despite the rather loose statements. But the reviewer was unable to disentangle on page 77 the rather obscure proof of the theorem that the cross-ratio of four tangents to a non-singular cubic from a point of the curve is constant.

It is possible that weaker students may derive considerable benefit from the book, especially from pp. 93–110 which contain solutions to over eighty problems taken mainly from examination papers of the University of London. In fact, one gets the impression that the book will assist students in acquiring proficiency in problem solving, although they may not understand the difficulties associated with the theory of plane algebraic curves or even be aware of their existence. T. J. WILLMORE

An introduction to deductive logic. By HUGHES LEBLANC. Pp. xii, 244. 38s. 1955. (John Wiley, New York ; Chapman & Hall, London)

This is perhaps the best of a number of recent elementary texts on modern logic, attractively written and showing evidence of very great care having been taken to make definitions sound and explanations intelligible and reliable.

Sentence and quantification logic are both presented in the first instance as systems of valid sentences and then as axiomatic systems. All provable sentences are shown to be valid and all valid sentences provable, (i.e. the completeness of the axiom system with respect to the intended interpretation is established). The proof of completeness follows the Gödel 1930 paper and not the method recently introduced by Henkin, which is probably less suitable for a first course in logic despite its air of simplicity.

Of other topics considered, a chapter is devoted to quantification logic with equality and the Boolean algebras of classes and relations, stopping short of set theory. There are also sections on many valued logics, intuitionist logic and Gentzen's theory of natural inference.

The application (on p. 67) of truth tables to quantification schemata like $(x)F(x) \supset (Ex)F(x)$ is misleading, for we cannot treat $(x)F(x)$ and $(Ex)F(x)$ as independent atomic propositions ; it is surely as wrong to take $(x)F(x)$, $(Ex)F(x)$ as independent as it would be to take p & q and q as independent to test the implication $(p \,\&\, q) \supset q$ by truth tables. One might in fact regard the truth table analysis as showing, not that there are quantificationally valid schemata which are not tautologies but rather that $(x)F(x)$ and $(Ex)F(x)$ are dependent.

The book is well printed. Two rather amusing misprints are (p. 23) " John is elected, John is elected$\supset$Mary is promoted, deduce Harry is promoted " and (p. 186) " the image of the class of printers with respect to the relation father of is the class of fathers of painters." (A list of misprints supplied by the author will be included in the review in the *Journal of Symbolic Logic.*)

The book ends with a set of exercises, some designed primarily to facilitate the passage from words to symbols and some to fill in details, or provide variants on the text. R. L. GOODSTEIN

Axiomatique intuitionniste sans négation de la géométrie projective. By N. Dequoy. Pp. 108. 1250 fr. 1955. Collection de logique mathématique, 6 (Gauthier-Villars, Paris)

This book is of considerable interest for a variety of reasons. It is the first to give an account of the new negationless intuitionistic logic introduced by G. F. C. Griss some nine years ago, and the account is very readable. The distinction between negationless and the older intuitionism is clearly drawn and the strength of the new logic is rendered equally evident. The application of this system of logic to found a system of projective geometry is a far-reaching undertaking which has been carried out with great skill and much evidence of care, and has produced a system which is remarkably free from the scars one expects the intuitionistic knife to leave.

The novelty of negationless logic lies in the duplication of relations. To equality is contrasted inequality as a separate concept, and to same is opposed different, each with its appropriate axioms.

The logical connectives are: entails $\rightarrow$, disjunction v, and conjunction &. Negation of course is wanting, so that $\rightarrow$ and v are independent. The logical axioms are (pp. 13–14):

$$p \,\&\, q \rightarrow q \,\&\, p,\ p \,\&\, q \rightarrow p,\ p \,\&\, q \rightarrow q,$$
$$(p \rightarrow q) \,\&\, (q \rightarrow r) \rightarrow (p \rightarrow r),\ (p \rightarrow q) \rightarrow (p \,\&\, r \rightarrow q \,\&\, r),$$
$$(p \rightarrow q) \,\&\, (r \rightarrow s) \rightarrow (p \,\&\, r \rightarrow q \,\&\, s),$$
$$(p \rightarrow r) \rightarrow (p \,\&\, q \rightarrow r),\ (p \rightarrow q) \,\&\, r \rightarrow (p \rightarrow q \,\&\, r)$$

(performing & before $\rightarrow$ when brackets are omitted), and

$$a \mathrm{v} b \rightarrow b \mathrm{v} a,\ (a \rightarrow b) \rightarrow (a \mathrm{v} c \rightarrow b \mathrm{v} c),$$
$$\{a \rightarrow (b \mathrm{v} c)\} \,\&\, (b \rightarrow d) \,\&\, (c \rightarrow e) \rightarrow \{a \rightarrow (d \mathrm{v} e)\}.$$

The difference between negationless and the older intuitionism may be illustrated by the following example. To prove that $\sqrt{2}$ is irrational intuitionism *with negation* accepts a proof that $p^2 = 2q^2$ leads to a contradiction as a proof that p^2 differs from $2q^2$. In negationless logic the inequality

$$| p^2 - 2q^2 | \geqslant 1$$

must be proved directly; for example, we can show that

$$| (2k+1)^2 - 2q^2 | = | 2r+1 | \geqslant 1, \text{ where } r = 2k^2 + 2k - q,$$

and

$$| (2k)^2 - 2(2l+1)^2 | = 2 \, | (2l+1)^2 - 2k^2 | \geqslant 2.$$

The axiom system (for plane projective geometry) accordingly includes axioms of separation. Denoting by $A \omega B$ the assertion that A and B are different points we find the axioms (p. 23)

$$A\omega B \rightarrow B\omega A,\ (\mathrm{V}C)\{B\omega C \rightarrow A\omega C\} \rightarrow (A = B),$$
$$A\omega B \rightarrow (C\omega A) \mathrm{v} (C\omega B).$$

The second of these axioms is always used to prove identity, and helps to reduce the extent to which theorems break up into special cases in finitist geometries (like the reviewer's logic-free system which dispenses not only with negation but with all logical connectives and operators).

R. L. Goodstein

The compleat strategyst. By J. D. Williams. Pp. xiii, 234. 38s. 1954. (McGraw-Hill)

A junior clerk in a bookshop might be tempted to class all books on the theory of games under " light entertainment " and in most cases he will be wrong. This book, however, is entitled to a place in that category. It introduces you, without mathematics and mostly without proofs, to the theory of two-person zero-sum games; only in passing are non-zero-sum games mentioned.

The author starts from two-by-two games and proceeds, in easy steps, to " four strategy games and larger ones " in Chapter 4. His style is so amusing that one regrets that it would be thought improper to write in the same light vein about, say, set topology or quantum mechanics. His examples are subtly absurd. There is a (quite irrelevant) reference to horses kicked to death by Prussians (*sic*), and an example called the sports kit (really about French

roulette) dealing with "two Muscovite guards—call them A and B, though these are not their names." We are also told that Portia's game with the suitors (*The Merchant of Venice*) is only worth a suitor's while if "he cherishes Portia at least three times as much as he deplores bachelorhood. This information does not make his choice easier, but it sharpens and clarifies the issue." Chapter 5, from which this quotation is taken, "has the characteristics of a catchall". It includes some devices for solving games, and hints at economic applications, too. The connection with Linear Programming is illustrated by Stigler's classical nutrition problem, disguised as that of a buyer in Dantzig's butcher shop. (In this name, and in others, the initiated will find an implied acknowledgement of help and assistance received by the author from his colleagues at the Rand Corporation.)

There are plenty of exercises with solutions, and a table of random numbers. There is, also, an index. As to solving games, in the technical sense, we are told to look always for a saddle point first (there is an intriguing picture of a mountain pass on which a couple have hoisted a flag to mark the compromise between his wish to camp at high and hers to camp at low altitude). If there is none, we should use a rule of thumb which, admittedly, works only for small games.

In spite of the apparently facetious style, the book contains sound theory, not at all on a superficial level. It is meant to reach a wide non-mathematical audience and is perfectly suited for this purpose. The mathematician, who is prepared to take proofs for granted, or to look for them elsewhere, will find here an ideal exposition of the basic ideas and of their application.

Type and print, and the amusing pictorial illustrations by Charles Satterfield, are excellent.

S. VAJDA

Die Idee der Riemannschen Fläche. By H. WEYL. 3rd edition. Pp. vii. 162. DM 22. 1955. (Teubner, Stuttgart)

Recent exciting developments in the transcendental theory of algebraic varieties have revived interest in the theory of algebraic functions, and for this reason the present is a most appropriate time for the publication of the third edition of Professor Weyl's classical book on the Riemann surface. The book has been largely re-written, though the approach has not been changed in any essentials. But the great advances that have been made in topology and other related theories since the second edition appeared in 1923 has necessitated an extensive revision of the details of many of the proofs, and the result is that the new edition has the air of a live piece of mathematics instead of being a museum piece, as too often happens when a new edition of one of the great works of mathematics is brought out. A preface describing the course of recent developments is written with all the vigour to be expected from the illustrious author, and helps to make the new edition as much of a necessity to all present-day workers as the earlier editions were to their predecessors.

W. V. D. HODGE

Formes extérieures et leurs applications. I. By W. SLEBODZINSKI. Pp. vi, 154. 1954. Monografje Matematyczne, 31. (Warsaw)

The acceptance of exterior forms as part of the standard equipment of a mathematician has led to the appearance of a number of books on the subject, but not so many as to leave no place for another one. The present book is the first of a pair of volumes which have grown out of a course of lectures delivered at the University of Wroclaw. It deals with the algebraic theory of forms, and the second volume which is promised will deal with the analytical theory.

The first chapter gives the necessary definitions and the principal general theorems, and this is followed by a chapter on algebraic exterior equations—a necessary preliminary to the study of Pfaff's problem which is to be considered

in the second volume. The symplectic group plays a fundamental role in the theory of exterior forms of the second degree, and the third chapter gives an account of some of the principal properties of this group, and of symplectic space. The fourth chapter deals with various applications of the theory of forms.

The full usefulness of this volume cannot be judged until the second volume has appeared, but it contains much that is valuable, and holds out the promise that the complete work will be an important addition to the literature.

W. V. D. HODGE

Vorlesungen über die Theorie der algebraischen Zahlen. By ERICH HECKE. Second Edition edited by Wilhelm Maak. Pp. viii, 266. DM. 11. 1954. (Akademische Verlagsgesellschaft, Leipzig)

This is a completely unaltered reprint of the original edition (1923), probably photographic, together with a short eulogy of the book by Maak. Hecke's book remains one of the best introductions to algebraic number-theory, and is particularly suited for those without a grounding in modern algebraic methods.

J. W. S. C.

Ricci-calculus. By J. A. SCHOUTEN. 2nd edition. Pp. xx, 516. DM 55; linen, DM 58.60. 1954. Grundlehern der mathematischen Wissenschaften, 10 (Springer, Berlin)

When *der Ricci- Kalkul* first appeared in 1923, tensor analysis was beginning to be accepted as a useful calculus, and differential geometry was expanding under the stimulus of general relativity. Since then there have been considerable developments in both the technique and applications of tensor analysis, and the task of bringing *der Ricci-Kalkul* up to date is certainly formidable. This the author has done, however, with great care and good judgement, and the present English edition is a remarkable *tour de force*. It is nowhere a mere translation of the first edition. Even the elementary treatment of tensor algebra and calculus has been changed in the light of recent work on the foundations of the subject, and in the later chapters there is much that is completely new.

The author has been faced with the difficulty of selection, for it would clearly be impossible for him to include in one volume all the many new applications of tensor analysis. As in the first edition, the author again restricts himself to geometrical applications, but even here there is an *embarras de richesse*; Riemannian and affine connected spaces of 30 years ago have been generalised in many ways, and there is now a great number of differential geometric structures to choose from. In making his choice the author evidently decided to restrict himself to the more immediate generalisations of Riemannian space and so to exclude all generalisations such as Finsler spaces, generalised path spaces, and the spaces of Cartan and Kawaguchi.

Another, and perhaps more serious, decision was to exclude virtually all topological considerations. This is justified because the book is primarily on tensor analysis, treated as " suffix calculus ", and therefore suitable only for the study of " local " geometry. The restriction is none the less serious because this book is most likely to be read by differential geometers, and for them the most exciting development in recent years has been in differential geometry in the large—the union of classical differential geometry with topology. For this subject it is desirable, of course, to know the local theories described in the volume under review, but it is also advisable to study, for example, the suffix-free calculus of Cartan and the modern topological definitions of geometrical objects involving the theory of fibre bundles.

To a certain extent, therefore, the scope of the present book is limited, but within its limitations it is as comprehensive and thorough as it could possibly be. The table of contents alone makes impressive reading, and the bibliography, occupying 87 pages, with something like 1400 entries, gives a most valuable record of research on geometrical applications of tensor analysis up to about 1952. This underlines a valuable feature of the book, which is that every section, however small, is provided with all the relevant references to original literature.

The first of the eight chapters into which the book is divided is on algebraic preliminaries and gives the usual algebra of tensors with the various notations now in use. In the second chapter, on analytic preliminaries, an n-dimensional manifold is defined together with its allowable coordinate systems, and tensors now occur as geometrical objects in this manifold. Differential operators and equations are discussed at some length and Pfaff's problem is considered, though only briefly, because the author, in collaboration with V. D. Kulk, has already written a treatise (Oxford, 1949) on this subject.

The third chapter is on linear connexions and gives all their classical properties. It is shown why such a structure is needed and how it leads to the concepts of parallelism and curvature and to the definitions of geodesics and normal coordinate systems. These general results are illustrated very nicely in the next chapter, on Lie groups; it is mainly confined to local properties, expressible in terms of a single coordinate system, and examines certain connexions which are associated naturally with a Lie group.

Chapter V is on embedding and curvature and is concerned with the formal relations between the structural tensors of a space and a sub-space. These include the generalisation, to Riemannian sub-spaces, of the classical Serret-Frenet formulae for a curve.

The next chapter is on projective and conformal transformations of connexions, and after that comes a chapter on variations and deformations. This includes an interesting account of groups of motions of a given space and of the holonomy groups associated with the various connexions. Finally there is a chapter on miscellaneous examples, including fairly full accounts of harmonic and complex spaces, and brief accounts of others, such as spaces of recurrent curvature.

The above description does little more than give the chapter headings and certainly does not do justice to this book, with its vast collection of ideas and results taken from a great number of original papers. As might be expected the style is too condensed for the book to be suitable as a text for the ordinary student, but for the specialist it will be invaluable as a book of reference. Great care has been taken over questions of notation and the choice of type—an important matter when suffixes play such an important part—and the printing and production generally are excellent. A. G. WALKER

Geist und Wirklichkeit. By K. REIDEMEISTER. Pp. iii, 92. DM 8.60. 1953. (Springer, Berlin).

Die Unsachlichkeit des Existentialismus. By K. REIDEMEISTER. Pp. iv, 40. DM 4.80. 1954. (Springer, Berlin)

Geist und Wirklichkeit is divided into two parts. Pages 3–41 are devoted to nine short essays on themes dealt with more systematically in the second part of the book (pp. 42–92) entitled " Prolegomena einer kritischen Philosophie ", which consists of an enterprising attempt towards an epistemological re-orientation inspired by Hilbert's metamathematics.

Die Unsachlichkeit des Existentialismus is a critical discussion of existentialism and positivism in the light of the author's philosophy as presented in *Geist und Wirklichkeit.*

Of special interest to the reader of this journal will be the author's views concerning mathematics and science, of which the following is a short résumé.

A distinction is made between *thinking* (Denken) and reason (richtiges Denken), i.e. thinking in accordance with a preassigned logic. Thinking is a process of testing endowed with the consciousness of self-certainty as epitomised by Descartes' *cogito ergo sum*. Testing involves comparing and hence the iterability of equals, and finds its objectification in the combinatorics of written or spoken signs which form the basis of metamathematics. Thinking as reflected in metamathematics is capable of attaining to knowledge proceeding from self-certainty independent of any other cognitive relation. It is an autonomous activity prior to reason. This is the fundamental ontological principle of the author's philosophy in support of which it is claimed that it represents the only conception of thinking reconcilable with the logical pluralism of metamathematics. Dogmatic rationalism (i.e. apriorism of reason) is criticised for failing on this score, and, moreover, for leading to self-contradiction. (The antinomies of set-theory are directly ascribed to the rationalistic attitude).

The scientific character of mathematics is established by way of metamathematical interpretation. Indeed the states of affairs of which Euclidean geometry speaks are not, by virtue of their universality, matters of fact. Nor does Kantian pure intuition of space offer them an adequate phenomenological alibi. For the *a priori* of pure intuition, so far from having been explicitly exhibited, has itself been deduced from the formal nature of geometry. Mathematical states of affairs are only attainable by thinking. To wit, the validity of a mathematical assertion is only evinced by proof, and the author concludes that the truth of mathematical propositions is determined by the logical structure of the system to which they belong.

The reviewer finds this last conclusion doubtful in view of Gödel's incompleteness theorems. If the author has a formalist conception of mathematics, terms like " truth ", " states of affairs ", etc., seem to be out of place ; or, perhaps, " mathematics " is intended to refer to completely formalisable theories only. At all events this point appears to be sufficiently crucial to merit more explicit treatment.

In his discussion of natural science the author maintains in opposition to orthodox positivism that scientific statements are not verified individually. Just as mathematics is concerned with formal systems, so science investigates real structures of interactions (Wirkungszusammenhänge), i.e. systems of properties and relations of observations. The properties and relations themselves are not subject to observation. The validity of a scientific theory consists in the logical isomorphism between a formal system and a structure of interactions. A scientific statement is said to be true if it has the same logical structure as a given (real) state of affairs. A structure is real if it is constant (beständig) ; and constancy is a matter of experience. " . . . dass also die Gesetzlichkeit des Wirkungszusammenhanges ein Zug der Natur ist, den wir nicht nach Prinzipien erkennen, sondern entdecken." All discussion concerning methodology is relegated to the province of scientific activity itself.

The presentation which is addressed to the general reader is non-technical throughout, and the reviewer feels that this makes a definitive evaluation of the author's ideas premature. A more formal exposition should be an undertaking of considerable interest.

M. H. Löb

Transfinite Zahlen. By H. Bachmann. Pp. vii, 204. DM 29.80. 1955. Ergebnisse der Mathematik, New Series, 1. (Springer, Berlin)

This is a very welcome first volume in the new series of " Ergebnisse der

Mathematik ". It is a book for specialists in the field, not for beginners, written in a crisp condensed style, packed with detail and up-to-the-minute information. The book covers roughly the same ground as Fraenkel's recent *Abstract set theory*, but deals more thoroughly and extensively with ordinal arithmetic. There is for instance an account of Finsler's transfinite series of arithmetical operations and of extensions of Fermat's and Goldbach's problems to transfinite ordinals, and a chapter on unattainable ordinals and cardinals.

A feature of the book is the careful segregation of the consequences of the multiplicative axiom from results obtainable without its aid. In the ordinal theory there are isolated carefully noted occasional uses of the axiom; the account of cardinal arithmetic (where the part played by the axiom is so much more striking and important) is divided into two parts, the first concerning itself only with the more difficult (and subtle) theory without the multiplicative axiom and the second part making free use of the axiom and the Aleph hypothesis.

Although the book opens with a brief account of the paradoxes of set theory and declares its allegiance to the Zermelo-Fraenkel type of set theory, foundations questions are not entered upon. This concentration upon arithmetical rather than logical properties leads to the exclusion of some of the most interesting and important recent applications of transfinite ordinals, for instance Gentzen's reduction of transfinite induction for ordinals less than the first ϵ number to ordinary induction, and the proof that this reduction is not possible for induction over ordinals including ϵ.

R. L. Goodstein

Numerische Behandlung von Differentialgleichungen. By L. Collatz. 2nd edition. Pp. xv, 526. DM 56; linen, DM 59.60. 1955. Grundlehren der mathematischen Wissenschaften, 60 (Springer, Berlin)

This comprehensive work, now in its second edition, exhibits the true vein of thorough and painstaking German scholarship. Carl Runge (1856–1927), to whom this volume owes so much, was a pioneer in the systematic study of numerical processes, though one should not forget that the Adams-Bashforth method, still reckoned to be probably the best of its class, dates back to 1883. Too great an emphasis on the explicit solution of elementary differential equations can mislead both the young mathematician and the young engineer; for the latter, at any rate, modern texts provide a corrective by paying some attention to the graphical and numerical processes which he will be obliged to use in technical applications.

Collatz surveys the whole field of differential equations. After a preliminary chapter on analytical machinery and general principles, there are sections on initial and boundary value problems for ordinary and partial differential equations, and a final chapter on integral and functional equations, where many questions still call for consideration. Many illustrative examples are fully worked out in the text, and there is a reasonable supply of exercises for the student, with solutions; the author must have put a great deal of hard work into this part of the book, work not immediately rewarding but of very great ultimate value to the conscientious student. One particularly commendable feature is the careful and extensive documentation, by use of which it should be easily possible to make further study of special points in the original memoirs. Biographical notes are appended to some names, and a noble photograph of Runge makes an admirable frontispiece.

The novice would be well advised not to plunge at once into these 500 closely-packed pages, but to begin by making a general survey of the field in some less detailed text, such as those by W. E. Milne, *Numerical solution of differential equations* (1953) and A. D. Booth, *Numerical methods* (1955). Thus

equipped, he will then find Collatz an indispensable and encyclopedic work of reference giving information about almost every method he is ever likely to require. The complicated typographical problems presented by such a work as this have been solved by the firm of Springer in their usual confident and efficient manner. In all, this is a worthy member of a famous series, and must find a place in any good scientific library. T. A. A. BROADBENT

Begründung der Funktionentheorie auf alten und neuen Wegen. By L. HEFFTER. Pp. viii, 63. DM 12.60. 1955. (Springer, Berlin)

This critique of fundamental concepts in the theory of functions is concerned with the conditions to be imposed on $f(z)$ if this function is to be " analytic ", that is, to have a power series expansion. Goursat showed that *continuity* of $f'(z)$ is not needed in establishing Cauchy's theorem (that the integral of $f(z)$ round a simple closed curve is zero), Osgood and Morera examined the consequences of assuming only that $f(z)$ obeys Cauchy's theorem, Looman and Menchoff obtained very precise conditions to be imposed on the partial derivatives satisfying the Cauchy-Riemann equations. The author himself has added to our knowledge by studying what may well be called Cauchy-Riemann *difference* equations. A text-book is likely to adopt one line of approach and so a supplementary work which is comparative in function is bound to be valuable both to the teacher and to the novice starting to make a serious study of function-theory. In this type of work, precision is all-important, and so the first 30 pages give a brief but clear statement of the preliminaries on series, differentiability and complex integration. A concise table of comparison makes the relations of the various approaches evident, and there is a select and annotated bibliography. T. A. A. B.

Dynamics of a particle introduced via the calculus. By P. W. NORRIS and W. S. LEGGE. Pp. 80. 4s. 1955. (Cleaver-Hume)

This is a separate—and commendably cheap—reprint of the first four chapters of the wellknown *Mechanics via the calculus* by Norris and Legge, now in its third edition. Linear motion and motion in two dimensions are dealt with in detail adequate for the scholarship candidate. T. A. A. B.

Higher transcendental functions. III. Edited by A. ERDÉLYI and the Bateman Project Staff. Pp. xvii, 292. 49s. 1955. (McGraw-Hill).

This volume completes that part of the Bateman Project concerned with special functions ; in content it goes somewhat beyond the general stock-in-trade of the working mathematician, as the chapter headings show : XIV, Automorphic functions ; XV, Lamé functions ; XVI, Mathieu, spheroidal and ellipsoidal wave functions ; XVII, Functions of number theory ; XVIII, Miscellaneous functions ; XIX, Generating functions.

Professor Erdélyi says that XIV and XVII are " frankly experimental ". A reasonably equipped mathematician who wanted a quick look at automorphic functions would find XIV helpful, though naturally the novice would find so concentrated a survey meat much too strong. But the number theory chapter seems neither a survey nor a simple collection of formulae, and so falls between two stools. This is probably inevitable, since number theory is surely *sui generis* ; nevertheless, the chapter with its select bibliography will supplement the indispensable Dickson. The chapter on Mathieu functions should do something towards organising that sprawling domain ; the compilers have wisely leaned heavily on McLachlan's standard treatise, and have also come down on the side of the Anglo-American notation.

Personally, I found the chapter on generating functions very much to my

taste. Much useful and interesting material, not easily to be collected from a multitude of sources, is here gathered together into a well-organised, clear and informative section. Incidentally, the sign $\doteqdot$ is used in symbolic identities such as that for the Bernoulli polynomials and numbers,

$$B_n(x) \doteqdot (x+B)^n,$$

and this is said to be " following Rainville (1946) " ; the symbol was defined and used very effectively for this general purpose by Milne-Thomson in 1933.

That the whole set of volumes forms an indispensable item for any scientific library need hardly be said ; clearly every mathematician needing ready reference to special functions must have these books at hand. We look forward to the completion of the vast undertaking by the publication of the second of the two volumes on integral transforms.

T. A. A. B.

Variable stars and galactic structure. By CECELIA PAYNE-GAPOSCHKIN. Pp. xii, 116. 18s. 1954. (Athlone Press, University of London)

Based upon lectures given in the University in 1952, this is the first astronomical work to bear the imprint of the Athlone Press of the University of London. It sets an exceedingly high standard of scientific distinction, of literary presentation and of typographical production for the long succession of others that one hopes to see following it. Books on observational astronomy are not normally reviewed in these pages ; so it must suffice to state that this one comprises a masterly survey, by a leading worker, of all that is known about the observational characteristics and the distribution and motions of variable stars in our own and neighbouring galaxies. It will be indispensable for the theoretical investigations that may be expected to be more fully noticed here in due course.

W. H. MCCREA

Cours de Géométrie Infinitésimale. By GASTON JULIA. 2nd edition, 1955. (Gauthier-Villars, Paris)

Second Fascicule : Cinématique et Géométrie Cinématique : Généralités : Chapters II–V. Pp. 1–80. 1500 fr.

Fourth Fascicule : Cinématique et Géométrie Cinématique : Étude approfundie du mouvement d'un corps solide : Chapters XII–XIV. Pp. 1–88. 1600 fr.

The first fascicule of this work containing Chapter I was reviewed in the *Gazette*, 1954, p. 217. In the succeeding fascicules the high standards of clarity in exposition which characterised the first chapter have been maintained.

The second fascicule begins with Chapter II which is concerned with the kinematics of a point. In Chapter III particular motions as translations, rotations and helicoidal motions of a rigid body are first examined, and this is followed by a discussion of the general motion of a solid body. The composition of motions is considered in detail in Chapter IV, together with kinematic and geometric applications. Chapter V deals with the problem of determining the finite motion of a solid body when the kinematic configuration is known at each instant. Here the student is introduced to the method of the moving frame of reference.

The third fascicule containing Chapters VI–XI has not been received for review.

The fourth fascicule studies the motion of a solid body in greater detail. Chapter XI deals in considerable detail with such topics as the instantaneous centre of rotation, and epicyclic motion. The motion of a solid body in which one point remains fixed is dealt with in the short chapter XIII which contains only six pages. Chapter XIV concludes the fascicule with an examination of the most general motion of a solid body.

The whole work proceeds in a very leisurely fashion, and can be read with little effort but with considerable enjoyment. The printing is good, and the numerous illustrations clarify even more the lucid description in the text.

T. J. WILLMORE

Deux Esquisses de Logique. By J. BARKLEY ROSSER. Pp. 70. 900 fr. 1955. (Gauthier-Villars, Paris)

This little book contains five lectures given by the author in Paris last year, and translated into French by Roger Martin. Although their material is more readily accessible to the English-speaking reader than the French the interest in the book will not be confined to France alone, for lucid accounts of fundamental ideas are all too rare in any language, in any branch of mathematics.

The topics considered are combinatorial logic and Church's calculus of λ-conversion, the notion of a model of a formal system, the Löwenhein-Skolem theorem and formalised set theory.

The conversion calculus treats of symbols of the form " $\lambda x \mid M$ " which are " applied " to terms. $\lambda x \mid M$ denotes " M as a function of x ", and $(\lambda x \mid M)N$ denotes the result of substituting " N " for " x " in the expression M ; if M' is the expression which results from this substitution we say that $(\lambda x \mid M)N$ is convertible into M'. Thus, for instance,

$$(\lambda x \mid x)N \text{ conv } N$$

where " conv " is of course an abbreviation for " convertible ", and

$$(\lambda x \mid M)x \text{ conv } M.$$

Where more than one variable is introduced a contracted notation is employed ; for example, $\lambda x \mid (\lambda y \mid M)$ is written as $\lambda xy \mid M$ and

$$((\lambda x \mid (\lambda y \mid M))P)Q \quad \text{as} \quad (\lambda xy \mid M)PQ.$$

It follows that

$$(\lambda xy \mid M)xy \text{ conv } (\lambda y \mid M)y \text{ conv } M.$$

The natural numbers 1, 2, 3, ... are represented in the calculus by the formulae

$$\lambda fx \mid (fx), \quad \lambda fx \mid (f(fx)), \quad \lambda fx \mid (f(f(fx))), \ldots,$$

and the successor function $\mathscr{S}$ is represented by

$$\lambda nfx \mid (f(nfx)).$$

To illustrate the operation of the calculus we consider the conversion of $\mathscr{S}1$ into 2.

Since $\qquad (\lambda fx \mid (fx))fx \text{ conv } fx,$

therefore $\qquad (\lambda n \mid (nfx))(\lambda fx \mid (fx)) \text{ conv } fx$

and so $\qquad (\lambda nfx \mid (f(nfx)))(\lambda fx \mid (fx)) \text{ conv } \lambda fx \mid (f(fx)),$

as required. Representing the sum of p and q by $p\ S\ q$, as another illustration we consider the conversion of the sum of 3 and 2 into 5. First we form $3S$, that is, $(\lambda fx \mid (f(f(fx))))S$ which is convertible into $\lambda x \mid (S(S(Sx)))$ and so $3\ S\ 2$ is convertible into $(\lambda x \mid (S(S(Sx))))\ 2$ which is convertible in turn into $S(S(S\ 2)))$, $S(S\ 3)$, $S\ 4$ and finally 5.

The Löwenheim-Skolem theorem states that every consistent formal system has a denumerable model. Rosser's proof of this remarkable theorem is a simplification due to Hasenjaeger of a proof by Henkin. The proof proceeds by successive enlargements of the system ; first a sequence of constants $a_1, a_2, \ldots$ is added. Then all formulae of the type $(\exists x)F(x)$ are enumerated, the nth being denoted by $(\exists x_n)F_n(x_n)$, and a sequence $\alpha(n)$ is defined so that for all $i < n$, $\alpha(i) < \alpha(n)$ and $\alpha(n)$ exceeds the indices of all constants a_j in all formulae $(\exists x_i)F_i(x_i)$ with $i \leqslant n$. The next enlargement adds all the formulae

$$[(\exists x_i)F_i(x_i)] \rightarrow F_i(a_{\alpha(i)})$$

as axioms and at the last stage we add any formula (without free variables) as an axiom if neither the formula itself, nor its negation, is provable in the

system thus far attained. The enlargements are shown to leave consistent a consistent system, and in the resulting system a formula $(x)F(x)$ is provable if and only if $F(a_i)$ is provable for every i. From this it readily follows that there is a model of the system with the denumerable sequence a_i as elements.

The final chapter is concerned with non-standard models of class-logic, that is, models which are not isomorphic with the intended interpretation of the logic, and with the use of non-standard models to prove the independence of axioms. R. L. GOODSTEIN

Théorie Métamathématique des Idéaux. By A. ROBINSON. Pp. 186. 2,400 fr. 1955. (Gauthier-Villars, Paris)

The aim of this book, like that of the author's previous work *On the Metamathematics of Algebra* is to show the important part which the study of formal systems can play in the generalisation and proof of theorems in modern algebra. Despite the similarity of aim and title the two books differ considerably in their contents and the present work, which is of the same very high standard as the first, may (in a sense) be called a fulfillment of much which was promised in the earlier volume.

Théorie métamathématique des idéaux ranges over a wide field of mathematical concepts and techniques. The book opens with an account of topological and metric spaces, ordered and partially ordered sets ; this is followed by a study of the theory of logical systems, based on the Hilbert-Bernays axioms for the propositional calculus (in axiom 3.2.2 the first x should be replaced by y), the notion of a Tarski-system playing a fundamental part. A sub-set S of the set of all propositions is called a Tarski-system, or system-T, if S contains all true propositions and all consequences of propositions in S. A system-T which does not contain the totality of propositions is called a proper system-T, and one which is not a proper part of another system-T is said to be maximal. The class Σ of proper systems is shown to be partially ordered by the relation of inclusion, and to have a least upper bound ; a theorem of Lindenbaum that every proper system-T is contained in a maximal system then follows by an application of Zorn's lemma. A second proof of this important result, without using the lemma, is given later in the book (p. 82).

The central concept of the book, a metamathematical ideal, is defined as follows. Given two sets of propositions K and I_0, then any set of propositions I is called an ideal in the domain I_0 with respect to K, or more precisely a metamathematical ideal, if I contains all the propositions of I_0 which are entailed by propositions in K or in I itself. Taking I_0 to be the class of all statements of the form $a=0$ for the constants a of a commutative ring Σ and taking K to consist of the axioms of the ring and the " positive " relations between the elements of the ring (i.e. true statements into which negation does not enter), if I is a metamathematical ideal in this domain I_0 with respect to this class K and if I^* is a sub-class of the constants of the ring such that $a \epsilon I^*$ if and only if $a=0$ is a statement in I, then I^* is called a *transformed ideal*. It is proved that the transformed ideals are precisely the algebraic ideals of the ring Σ.

Amongst many interesting applications is a study of Ritt's differential algebra in which Ritt's results are derived directly from the general theory of metamathematical ideals.

The book concludes with some important examples of a quite different way in which metamathematics may be applied to algebra. Instead of proving particular theorems about special structures or even about all structures which satisfy some given set of axioms one seeks to prove that all statements of a certain kind which are true for certain particular structures are true also for certain other structures. For example the Ostrowski-Neother theorem on irreducible polynomials is derived from the following result : Let Σ be a finite

algebraic extension of the field of rationals and Σ' the ring of algebraic integers in Σ; further let X be a statement in a formal language L, expressed by means of the constants and relations of the ring, which is verified by all commutative field extensions of Σ. Then X is verified also by all commutative field extensions of the quotient rings Σ'/I for all but a finite number of prime ideals I of the ring Σ'.

R. L. GOODSTEIN

Mathematics and plausible reasoning. I. Induction and analogy in mathematics. Pp. xvi, 280. 42s. **II. Patterns of plausible inference.** Pp. x, 190. 35s. By G. POLYA. The set, 70s. 1954. (Princeton University Press; Geoffrey Cumberlege, London)

I hope that no teacher will allow himself to be deterred from reading these volumes by the feeling that the title is high-falutin and that therefore the contents will be beyond his scope. Volume I should be read by every sixth-form teacher and by every mathematical specialist, either in the sixth form or in his undergraduate days. If we teachers wish to help our brightest pupils in their first attempts at research, we are not doing enough if we merely drill them in problems of the "Given A, prove B" type; they must also learn how to frame questions and how to guess the probable answers. Perhaps this attitude of mind can not be taught; but there is no reason why it should not be acquired and developed. As Polya remarks, a serious student of mathematics must learn demonstrative reasoning, the distinctive mark of his subject, but he must also learn plausible reasoning, that is, he must know how to make good guesses, for on this his creative work will depend. There is no royal road to this end, but Polya offers us much sound comment, many examples for imitation and ample opportunity for practice. We start with simple ideas about numerical and geometrical induction, but very soon we are seeing how some of the striking theorems of mathematics *might* have been discovered—not, of course, always *how* they were discovered, and not often how strict logical demonstrations can be constructed. The richness of the volume can hardly be conveyed in a short notice, but I may mention two of my own favourites. First, a beautiful chain of plausible argument leading to the conjecture (which happens to be correct) that if u is an odd number, then the number of ways in which $4u$ can be expressed as a sum of four odd squares is equal to the sum of the divisors of u. Secondly, from Vol. II, the proof that for positive numbers $a_1, a_2, \ldots$ (and sums to infinity)

$$\Sigma(a_1 a_2 \ldots a_n)^{1/n} < e\Sigma a_n.$$

I remember reading Polya's proof of this theorem (due to Carleman) in the *Proc. London Math. Soc.* in 1925, and being dazzled by the ingenuity of the artifice employed. Now I see that the word "artifice" does an injustice to the controlled guess-work and ability to learn from apparent failure on which the argument depends. A bright boy at scholarship level would surely take to this book as a duck takes to water, and would profit from every page and particularly from the rich collection of exercises. Hints on a generous scale are supplied at the end of the book.

Vol. II is hardly less interesting than Vol. I, but it is more difficult to read and its appeal is largely to the sophisticated reader. Suppose $P(n)$ is a conjecture which we hope may be true for all integers n; we know that it is true for certain n, and we now consider it for some other value of n, say v. If $P(v)$ is false, the test is decisive; but if $P(v)$ is true, what have we learned? Is there any ground for asserting that the conjecture is now more probable, and is any quantitative estimate of this increase possible? Clearly here we are in the domain, not only of probability, but of psychology. The difference $\pi(x) - \mathrm{li}\,(x)$ is negative for all x up to 10^7; would it not be fascinating to know

what line of conjecture, presumably in opposition to the available numerical evidence, led Littlewood first to suppose, and then to prove, that somewhere or other $\pi(x) - \mathrm{li}(x)$ is positive? The material displayed in Vol. I is used by Polya to provide data for his very thorough discussion of this thorny field of inductive probability.

Both volumes provoke thought and stimulate the imagination; the first volume, if not the second, should find its way into all school libraries.

T. A. A. B

Contributions to the theory of partial differential equations. Pp. v, 257. 32s. 1955. *Annals of Mathematics* studies, 33. (Princeton University Press; Geoffrey Cumberlege, London)

Like some other volumes in this series, No. 33 contains research memoirs, fifteen in all, presented to a three-day conference on the topic of partial differential equations, and forms a welcome addition to the few works available on this rather neglected branch of analysis. D. H. PARSONS

Elementary statics. By A. E. SHORT. Pp. 342. 18s. 1955. (Geoffrey Cumberlege, Oxford University Press).

Written to meet the needs of candidates for University scholarships or the General degree of London University, this book is divided into three parts: (i) on forces in general, as represented by line vectors; (ii) on individual forces, classified in separate chapters as weight, reaction, tension and thrust, shearing stress and bending moment, gravitational attraction, fluid pressure; (iii) on work and energy, including the principle of virtual work, simple potential theory and stability. This arrangement is designed to limit the number of new conceptions in a problem of equilibrium; it is on the whole effective, though it separates the potential theory from attractions and entails, for instance, that problems in the chapter on reaction may not involve strings and pulleys. The chapter on weight is concerned with centres of gravity, that on reaction includes the usual work on friction, while under tension and thrust come frameworks and catenaries.

In the general theory of the first part there is a chapter on vectors which, as often happens, appears to be a conventional gesture rather than an introduction to a method to be used. Although the vector product is included and moment about a point defined, these are not related to the reduction of forces in three dimensions, of which there is an account in Ch. 4 in cartesian coordinates. Some space is saved by regarding parallel forces as a special case of any system, for which the principle of moments has been proved by reduction to a single force and a couple.

In the last part the extension of the principle of virtual work from a single particle to rigid bodies could be less abrupt. In the final statement there is no mention of order of small quantities though it is evident, in passing to the principle of stationary potential energy, that the work done in an actual small displacement is not strictly zero. Mention of the first order of small quantities is made in evaluating the work done by internal stresses, where it could be shown, with the usual assumptions of rigidity, that the work is strictly zero; it is the external forces of the body which do work of the second or higher order of small quantities.

In each chapter there are examples for the reader after suitable sections of theory or exposition and a plentiful supply of general questions at the end of the chapter. An omission noted is the equation $T = T_0 e^{\mu\psi}$ for a weightless string in limiting equilibrium on a rough curve; it could be made a corollary to Ex. (v) on p. 155. The book is well produced and no errors or misprints have been detected. C. G. PARADINE

Integers and theory of numbers. By A. A. Fraenkel. Pp. 102. $2.75. 1955. *Scripta Mathematica* Studies, 5 (Yeshiva University, New York)

This is the first of a series of monographs on modern mathematics, based on Fraenkel's talks in the Israel adult education programme; further volumes will deal with modern algebra and transfinite numbers. They are intended for the competent pupil or intelligent layman who may wish to know what modern mathematics is doing.

The present volume discusses the integers as cardinals and ordinals, then proceeds to theory of numbers (primes, Fermat's theorems, algebraic numbers and ideals) and ends with a formal abstract account of the extension from the field of integers to the field of rationals. It is gratifying to see how a skilled exposition can cover so much ground with very little technical apparatus, though inevitably some results are merely quoted, while in some instances the proofs are relegated to an appendix. But this ease should not delude the careful reader, who will mark Fraenkel's warning that the treasures of mathematics in this field " may be plucked only by one armed with the weapons of higher analysis and with the abstract and complex methods of modern arithmetic ". Some popularisations leave the impression that the recondite considerations so often encountered in the professional texts are merely smoke-screens thrown up by the mathematician in order to enhance the " mystery " of mathematics. Fraenkel is far too fine a mathematician to mislead earnest readers in this fashion, and while his book will no doubt stimulate interest in number theory, it will certainly not encourage the " mathematics without tears " nonsense. His message is plain enough: neither understanding nor accomplishment can be attained without hard work. His explanations and his results are attractive enough to make the prospect of hard work seem worth while.

T. A. A. B.

Tables of Functions and of Zeros of Functions. Pp. xi, 211. $2.25. 1954. Applied Mathematics Series, 37. (National Bureau of Standards, Washington)

This volume, sub-titled " Collected Short Tables of the National Bureau of Standards Computation Laboratory ", contains 18 items, of which 14 have been published previously in standard and easily accessible periodicals, and in some cases have also been sold in pamphlet form. It has been decided to meet continuing demand by reprinting in this collected form, which if slightly rough and ready is nevertheless very handy. The original pagination has been suppressed in favour of pagination of the present bound volume as a whole. The original reference is always given in some form (when necessary, by reproduction of a typescript addition to the original printed matter), but this form does not always include page numbers, so that anyone who wishes to give a full and proper reference to the original is not always spared the trouble of looking it up.

Of the other items, the most noteworthy relates to the functions

$$E_n(x) = \int_1^\infty e^{-xu} u^{-n}\, du.$$

Tables for $n = 0(1)20$ and $x = 0(\cdot 01)2(\cdot 1)10$ occupy fifty pages; the accompanying text is mainly by G. Placzek. Although issued in 1946 as a report of the National Research Council of Canada, Division of Atomic Energy, Chalk River, Ontario, this item cannot be said to have been easily accessible in this country. It has become a standard reference in works on astrophysics, and its publication in the volume under review will be heartily welcomed.

There are also a table of sines and cosines for radian arguments between 100 and 1000, brief " radix tables " by H. E. Salzer for finding natural logarithms to 25 decimals, and a 36-page table of $x^n/n!$.

A. Fletcher

Oxford Graded Arithmetic Practice. Book One: Addition. Book Two: Subtraction. By D. A. Holland. Pp. 64. 2s. each. Teachers' books. 2s. 6d. each. 1954. (Oxford University Press)

One of the aims given in the introduction is " to provide a complete analysis of addition and subtraction for step by step teaching." This constitutes one of the chief merits of these books. Skilfully graded work is of the utmost value in ensuring understanding and accuracy in the fundamental processes.

The format of the books in this series is excellent from the teacher's point of view—diagnostic and remedial assessments, and marking, are all facilitated by the logical grading and arrangement of examples. The inexperienced teacher, however, may not realize that the books are for practice only and, by failing to use them in conjunction with practical work, will defeat the intention of the writer.

From the pupil's angle the arrangement is probably less successful than from the teacher's . Illustrations and variety in some form, although the books are for practice only, are needed to keep the subject alive, particularly for the less able child.

In both books the number charts are useful. The number lists are a little unusual and are an important feature in stressing the comprehension of words and numbers. The books are suitable for practice throughout the Mental Age Range 7 to 11 +, with the younger children for consolidating the learning of the basic processes and number facts, with the older children for achieving speed.

R. E. M.

Practical mathematics. I. By C. C. T. Baker. Pp. vii, 253. 7s. 6d. 1955. (English Universities Press)

This textbook for use in Technical Secondary schools and preparatory courses of Technical colleges leaves very little for the teacher to explain, but the result is that much of the book contains many brief and peremptory questions of the drill type. They are very necessary, but none the less very tedious to look upon and are best provided by the teacher. It would be better to replace half of them by problems culled from the drawing office, workshop and laboratory.

The three chapters on geometry and the lead into trigonometry from similarity are excellent features while the introduction of elementary statics and hydrostatics greatly adds to the interest.

In Vol. II there is a very pleasing chapter of formula manipulation, but the exercises on indices and factors seem too long and tedious. The recommended method of computation by logarithms is not the best for facility or accuracy.

A. J. L. Avery

Technical mathematics. By H. S. Rice and R. M. Knight. Pp. xiv, 748. With answers, 52s. 1954. (McGraw-Hill)

The very serious attention of all Technical teachers is drawn to this book which certainly achieves all that it claims. It is thorough in treatment, practical in its outlook, and embraces the practical mathematics scope of every possible branch of engineering. It is a very encyclopedia and source book for teachers, and will be a great help to all engineering students. Those who find mathematics a stumbling block will appreciate the lucid explanations and the orderly arrangement. It is very modern in outlook and valuable chapters on the full scope of graphs, vectors, periodic motion and complex numbers are included. It is a book which is a delight to read and use, the paper, print, diagrams and arrangements being perfect.

Unfortunately mathematics is here a tool and only a tool. No British course going so far or so deeply would fail to include the calculus.

A. J. L. Avery

Fundamental number teaching. By R. K. and M. I. R. POLKINGHORNE. Pp. 196. 8s. 6d. 1955. (Harrap)

The authors have written this book at the request of teachers who have read their articles on Primary School Arithmetic in the *Teacher's World*. It deals with all the arithmetic needed by even the most able of junior school children and includes two very useful chapters on vulgar fractions and one on decimal fractions. Both the practising teacher and the student in training should find this a very useful reference book. Not only does it deal fully with the processes of arithmetic but there are at all stages helpful suggestions on teaching methods. The book is written very simply and there are many references to actual experiences with children.

The importance of the recording of practical work and the practice which it gives in the use of language is recognised by the authors but could have been enlarged upon. Scale is mentioned but there is no discussion of its fundamental principle of ratio nor of ways of introducing it.

The necessity for a sound understanding of notation and of the " way numbers behave " is stressed throughout the book and there are many practical suggestions of ways in which children can be helped to develop a number sense and become " at home with numbers ".

Perhaps the least satisfactory part is those chapters which deal with the compound quantities; money, weight, time, etc. Although the need for practical experience is recognised, the average teacher who wants to give the children this experience and who comes to this book for advice will find little help in the problem of how to organise the work of a large class. 118 pages are devoted to notation and the four rules of number, and only 35 to all those compound quantities which make up the greater part of a junior school syllabus. They surely need and deserve a fuller treatment.

K. SOWDEN

Mental arithmetic tests, 1-5. By N. J. FILMORE. 1s. each. 1955. (A. & C. Black)

This is a series of five books. Each book contains 30 tests of 15 questions. They are attractively simple in cover and lay-out, and large clear print is used. The books are well graded, beginning with work for young juniors, and by the time Book 5 is reached most of the measures and processes dealt with in the junior school have been introduced. Teachers of brighter-than-the-average children will perhaps find that the speed is too slow in the middle of the series and that another book is needed giving the kind of mental arithmetic examples set in so many selection tests.

The exercises are both mechanical and problem, the latter using realistic material within the experience of the children. It would be a considerable help to the teacher if each book contained a list of new processes or measures to be introduced.

K. SOWDEN

Vital primary arithmetic. IV. By R. S. WILLIAMSON. Pp. 96. 3s. 6d. 1954. (Macmillan)

This is the fourth book of a series for junior school pupils. It is also offered for first-year secondary modern pupils. It is packed with examples both mechanical and problem. It would perhaps be a more satisfactory book if it had been less crowded. The economy in the use of words and of space has resulted in a lack of clarity in some places.

A considerable amount of material of a realistic and useful nature has been used but the problems are not always equally realistic or useful: for example, " To make the Mersey tunnel rock was dug out at a rate of 262,080 tons a year (52 weeks). How much a minute? " There is considerable breadth of application of mathematics, occasions for its use being found in Music,

Literature, Nature History and Geography, and the interested teacher will find many ideas for the integration of mathematics with other subjects. The sections on geometry are inclined to be formal and if meant as a first introduction to the subject are not very likely to arouse enthusiasm in the average pupil. K. SOWDEN

Modern School mathematics. I. By E. J. JAMES. Pp. 132. 5s. 1955. (Geoffrey Cumberlege, Oxford University Press)

This is the first book of a series of four for secondary modern school pupils. To quote Mr. James : " The work in the four books follows a series of courses each of which should last 2–4 weeks ". The book contains courses on notation, scale, representation, tabulation of numerical material, angle measurement, fractions, area, decimals, averages and percentages. This sounds a somewhat formidable course for one year but the calculations are in all cases very simple and the work is largely practical or has simple realistic applications. For example, addition and subtraction of fractions are led up to through the principle of equivalence so that the finding of the common denominator is seen as a simplification not as a magic ritual. One feels that by the end of the year the pupils should have gained a real understanding of these fundamental ideas of mathematics and should be able to proceed with confidence to their development in the second year.

The first course in Book I covers 15 pages and is designed to arouse interest and curiosity about the way numbers behave. It includes magic squares, series and examples leading to surprising answers such as those obtained by multiplying 37 by multiples of 3. It should prove a stimulating new approach for those children to whom arithmetic has been merely " sums ".

The whole book shows, not only the ways in which needs for calculations arise but the presence of mathematics in so much that is around us ; finding our way from a map, how to place the furniture in our new house, the growth of a bean plant, the shape of church windows and the varying length of the sun's shadow, to give but a few examples.

No calculations are included merely for their own sake, each has its realistic setting but the need for practice of processes in order to develop speed and accuracy is recognised by the insertion of a section at the end of the book giving over one thousand mechanical examples. K. SOWDEN

Mathematics for Higher National Certificate. I. By S. W. BELL and H. MATLEY. Pp. 293. 15s. 1955. (Cambridge University Press)

As the title suggests and readers of the Association's *Report on the teaching of mathematics in technical colleges* will expect, the authors of this book confine themselves to a concise treatment of the essentials of a fairly wide syllabus. Differentiation of e^x and $\log x$ are assumed as a starting point. After chapters on differentiation and simple integration there are separate chapters on hyperbolic functions and inverse functions before further integration and applications. Then follow curvature, convergence, standard series, the catenary, complex numbers, partial differentiation, differential equations and harmonic analysis. This last chapter has been included for the benefit of those students who will not take mathematics in the second year of their course. In comparison with the syllabus (A1) suggested in the report, this book goes further in including harder integrals such as that of $(px+q)/\sqrt{(ax^2+bx+c)}$ but not so far in differential equations, which stop at the homogeneous equation of the second order with constant coefficients. There are appendices on curve tracing and the coordinate geometry of the straight line and circle.

Some details of method may be mentioned. The mean value theorem is illustrated and later used in obtaining the total increment of a function of two variables. In appropriate standard integrals the constant of integration is

incorporated in the form log $\{Cf(x)\}$. Engineer's rules, in terms of the forces to the right of a section, are given for evaluating shear force and bending moment; in a text-book of mathematics one would prefer to see reasons for the rules, in terms of the equilibrium of the portion of the beam to the left. A few of the diagrams are not very good, particularly the cycloid and a curly squiggle representing successive partial sums of an alternating series. Additional diagrams to illustrate total increment and Newton's method of approximation would be an advantage.

Examples, both worked and set as exercises, are satisfactory and in general the book should serve well for the purpose for which it was written.

C. G. P.

Parmi Les Belles Figures de la Géométrie dans L'espace. (Géométrie du Tétraèdre). By VICTOR THÉBAULT. Pp. xvi, 287. 2000 fr. 1955. (Librairie Vuibert, Paris)

Members of the Mathematical Association who have enjoyed M. Thébault's articles in the *Gazette* will find much to interest them in his latest book, which gives a lucid account of the various geometrical properties of the tetrahedron. As the author points out, his aim has been to collect and make available diverse results which would otherwise have to be searched for in a large number of mathematical journals.

The four chapters of the book are Configurations fundamentals, Sphères associées à un tétraèdre et à un polygone gauche, Compléments à la géométrie récente du tétraèdre (first and second Lemoine points and the spheres of Tücker, Adams, etc.) and Questions proposées sur la géométrie du Tétraèdre.

No one reading the book could fail to be impressed both by the immense number of known properties of the tetrahedron and by the elegance and completeness of M. Thébault's account of them. R. W.

Méthodes d'algèbre abstraite en géométrie algébrique. By P. SAMUEL. Pp. ix, 133. DM 23.60. 1955.

La géométrie des groupes classiques. By J. DIEUDONNÉ. Pp. vii, 115. DM 19.60. 1955. Ergebnisse der Mathematik, neue Folge, 4, 5. (Springer, Berlin)

These two books reflect, in complementary fashion, the way in which modern algebra and classical geometry have reacted upon one another. The titles are in themselves sufficiently indicative; on the one hand we have modern abstract algebraic techniques helping in the foundations of algebraic geometry, and on the other we see geometrical language and ideas providing the inspiration and driving power in a study on group theory.

In short tracts of this nature, which aim at presenting recent advances in a particular branch of mathematics, one naturally cannot expect too complete an exposition. Professor Samuel has the more difficult task, and he is compelled to assume a very considerable knowledge of abstract algebra. Because of this, the book includes a " Rappel Algébrique " which should at least be of service to the semi-initiated. The main purpose of the tract is to lead up to and present the author's theory of intersection multiplicities—including the case when the dimension of the intersection is higher than it should be (" composant excédentaire "). Besides this, the book should prove of value to the worker in algebraic geometry because it contains in a concise yet readable form much of the standard material of the subject.

In writing on the classical groups Professor Dieudonné has thэ advantage of dealing with a topic with which the general reader will be reasonably familiar, even though he may find himself faced with a general non-commutative field instead of the real or complex numbers. Moreover the geometric language and techniques, which the author employs throughout with great facility,

make this one of the most readable books the reviewer has come across. Within his 100 pages Professor Dieudonné has contrived to present a fascinating survey, collecting together the results of much recent research (in great degree his own) and giving it a unity which appears almost deceptively simple. The longer proofs tend to be sketched, but otherwise nothing is assumed of the reader. A very extensive bibliography testifies to the scope of the book and the vitality of the subject. M. F. ATIYAH

The Collected Works of George Abram Miller. Volume IV. Pp. xi, 458. \$7.50. 1955. (University of Illinois Press, Urbana, Illinois)

Since the publication of Volume III of these Collected Works, Professor Miller has died (in 1951) and two of the members of the original Committee, Professors R. D. Carmichael and A. B. Coble, have retired. Thus the responsibility for the present Volume IV has fallen to Professor H. R. Brahana, the remaining member of the Committee which was formed in 1933 to supervise the publication of Miller's work, and in particular to select those of his papers to be included. Volume IV contains ninety-eight papers all published in the years from 1916–29. As in previous volumes, all of Miller's technical contributions to the theory of groups in the period covered seem to have been included, while with a very few exceptions his numerous publications on more general subjects, elementary mathematics and the history of mathematics (amounting to 134 articles in this period) have been sacrificed. It would not be possible except at inordinate length to give an adequate idea of the contents of this latest volume. Most of the papers are short and are concerned with such topics as, to give examples, the determination of all the abstract groups of order 72, the Frattini subgroup of a finite group, the characteristic subgroups of a finite Abelian group. A large number are devoted to substitution groups, to which subject Miller made many useful contributions in this period. It is a little surprising that the paper " An overlooked infinite system of groups of order pq^2 " should have been included without any hint that the argument which it contains is incorrect. However, in general nothing but praise can be given to the way in which the present volume, like its predecessors, has been produced. It will be very useful to those interested in finite groups and permutation groups to have Miller's extensive output in a form which obviates the need to refer to scattered periodicals. P. HALL

THE MATHEMATICAL ASSOCIATION

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BOOKS FOR REVIEW

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